NON-LINEAR FUNCTIONALS OF THE PERIODOGRAM

GILLES FAY, ERIC MOULINES, AND PHILIPPE SOULIER

Abstract. A central limit theorem is stated for a wide class of triangular arrays of non-linear functionals of the periodogram of a stationary linear sequence. Those functionals may be singular and not-bounded. The proof of this result is based on Bartlett decomposition and an existing conterpart result for the periodogram of an i.i.d sequence, here taken to be the driving noise. The main contribution of this paper is to prove the asymptotic negligiblility of the remainder term from Bartlett decomposition, feasible under short dependence assumption. As it is highlighten by applications (to estimation of non-linear functionals of the spectral density, robust spectral estimation, local polynomial approximation and log-periodogram regression), this extends many results until then tied to Gaussian assumption.

1. Introduction

There is a wealth of applications in time-series analysis where a quantity of interest depends on a triangular array of non-linear function of the periodogram. Examples of such situation include: "direct" estimation of non-linear functions of the spectral density (such as the innovation variance; see e.g. Chen and Hannan, 1980), parameter estimation by log-periodogram regression (see e.g. Taniguchi 1979, 1991), non-parametric estimation of the spectral density by robust non-linear regression of the periodogram (see e.g. Von Sachs, 1994 and Janas and Von Sachs, 1995), local polynomial regression of the periodogram (see e.g. Kreutzberger and Fan 1998, and Fan and Gijbels, 1996 , section 6.6), or more recently, the estimation of the memory parameter for long-range dependent processes based on some form of log-periodogram regression (see e.g. Robinson 1995, Velasco 1999 and Moulines and Soulier 1997).

Key words and phrases. periodogram, linear process, central limit theorem, non-linear functionals, Bartlett decomposition.

Because of the very involved nature this type of statistics, most of the derivations up to now have been obtained under the additional assumption that the process is Gaussian, being based on general results on non-linear functional of (possibly) non-stationary Gaussian process (see e.g. Taqqu (1977), Arcones (1994)). However, these techniques are tailored to the Gaussian case, and do not extend to a wider setting.

A first step to weaken the Gaussian assumption has been taken by Chen and Hannan, which proved the consistency (but not the asymptotic normality) of the sample average of the log-periodogram evaluated at the Fourier frequencies of a non-Gaussian linear process. This work was based on early results on the periodogram of linear processes obtained by Bartlett (1955, see also Walker, 1965).

Some extensions of these results have been obtained by Von Sachs (1994) and Janas and Von Sachs (1995), motivated by the limiting theory of a class of "robustified" estimator of the spectral density, using Huberized regression estimator in the spectral domain. Janas and Von Sachs (1995) proved the consistency of the estimators of general non-linear functions of the spectral density for non-Gaussian linear process. Contrary to the work of Chen and Hannan (1980), the technique used by Janas and Von Sachs (1995) was based on Edgeworth expansion of triangular array of weakly dependent random variables, using results established earlier by Götze and Hipp (1983). Because of the very involved nature of the Edgeworth expansion in the dependent case, only the first and the second order moments can be evaluated; based on these results, Janas and Von Sachs (1995, section 3.4) proved the mean-square convergence of their robust spectral estimator. Note that the approach used by these authors is restricted to the case where the process is strongly mixing with geometrically decaying mixing coefficients.

The main objective of this paper is to derive the limiting distribution of triangular array of non-linear function of the periodogram of a linear process under mild technical conditions. Our approach extends the techniques presented in Chen and Hannan (1980). It is based on the Bartlett decomposition of the periodogram which relates the periodogram of the observations to the (fictitious) periodogram of the "driving" noise. Using this decomposition, the proof is in two almost independent steps. The first step consists in showing a central limit theorem for triangular array of non-linear functions of the periodogram of an indenpendent and identically distributed (i.i.d) sequence of random variables. This first step is carried out in a companion paper by Fay and Soulier (1999). The second step consists in showing that the remainder term in the Bartlett decomposition does not contribute to the limit. This step, which is essentially independent from the first one, is carried out in this paper under conditions on the filter coefficients that imply short-range dependence, but for non-linear functions which are possibly singular and non-bounded (such as $\phi(x) = \log(x)$).

The paper is organized as follow. In section 2, the main assumptions and results are stated. In section 3, several applications (involving non-linear functions of the periodogram) are outlined to illustrate our results. Bounds for the remainder terms in the Bartlett decomposition of the periodogram are presented in section B.

2. Assumptions and main results

Define the discrete Fourier transform and the periodogram of a stationary process Y as

$$d_n^Y(x) = (2\pi n)^{-1/2} \sum_{t=1}^{n} Y_t e^{itx} \quad \text{and} \quad I_n^Y(x) = |d_n^X(x)|^2.$$

The discrete Fourier transform and the periodogram are evaluated at the Fourier frequencies $x_k = 2\pi k/n$, $1 \le k \le \tilde{n} := [(n-1)/2]$, where $[a]$ denotes the integer part of a. Let m be a fixed integer. Following the procedure proposed by Robinson (1995), the frequency axis is divided in non overlapping segments of size m, and the periodogram is pooled over each segment. Let $K = [(n-m)/2m]$ (the dependency in n is omitted). For $k = 1, \cdots, K$, denote $J_k = \{m(k-1)+1, \cdots, mk\}$. Define the k-th ordinate of the pooled periodogram of $Y_1, \ldots, Y_n$ as the sum:

$$\bar{I}_{n,k}^Y = \sum_{l \in J_k} I_n^Y(x_l) \tag{1}$$

Since $\sum_{t=1}^n \exp(itx_i) = 0$ for $1 \le k < \tilde{n}$, the pooled periodogram is shift-invariant. Let $\mathcal{H}$ be the space of measurable functions ϕ such that for all $u \in \mathbb{R}$, $\mathbb{E}[\phi^2(|u\boldsymbol{\xi}|^2/2)] < \infty$, where $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_{2m})$ denote a $2m$-dimensional Gaussian vector and $|.|$ denotes the Euclidean norm.

For $\phi \in \mathcal{H}$, define

$$\gamma_m(\phi) = \mathbb{E}[\phi(|\boldsymbol{\xi}|^2/2)], \tag{2}$$

$$\gamma_m(\phi, u) = \mathbb{E}[\phi(u|\boldsymbol{\xi}|^2/2)]. \tag{3}$$

Many statistical applications require to prove a central limit theorem for weighted sums of non-linear functionals of the pooled-periodogram ordinates of the form

$$S_n(Y, \phi) = \sum_{k=1}^{K} \beta_{n,k}\{\phi(\bar{I}_{n,k}^Y / f_Y(y_k)) - \gamma_m(\phi)\}, \tag{4}$$

$$\tilde{S}_n(Y, \phi) = \sum_{k=1}^{K} \beta_{n,k}\{\phi(2\pi \bar{I}_{n,k}^Y) - \gamma_m(\phi, f_Y(y_k))\}, \tag{5}$$

where f_Y is the spectral density of the process Y, $(\beta_{n,k})$ is a triangular array of real numbers and y_k, $1 \leq k \leq K$ are frequencies related to the set of frequencies J_k. For instance, an often convenient choice is $y_k = (2k-1)\pi/2K$. Unless otherwise specified, we only assume that $2\pi m(k-1)/n \leq y_k \leq 2\pi mk/n$. If Y is a sequence of uncorrelated variables, then $S_n(Y, \phi)$ and $\tilde{S}_n(Y, \phi)$ coincide. When Y is Gaussian white noise, the variables $2\pi \bar{I}_{n,k}^Y$ are independent $\Gamma(m, 1)$ r.v. (*i.e.* half a central chi-square with $2m$ degrees of freedom). Thus $S_n(Y, \phi)$ converges to a normal random variable under Lindeberg-Levy's conditions (cf. Petrov, 1995). When Y is not a Gaussian white noise, the random variables $(\bar{I}_{n,k}^Y)$ are no longer independent, and no general result is available for non linear functionals of $\bar{I}_{n,k}^Y$. Recently, Fay and Soulier (1999) have proved a central limit theorem for $S_n(Y, \phi)$ when Y is a (non-Gaussian) white noise, under rather weak and easily checked conditions on the functional ϕ and the weights $(\beta_{n,k})_{1 \leq k \leq K}$. For a general process X, it seems quite hopeless to prove directly a central limit theorem for $S_n(X, \phi)$ (cf. for instance Janas and Von Sachs (1995) for an attempt in that direction). If X is a linear process, that is a process which admits a linear representation with respect to a white noise Z :

$$X_t = \sum_{j=-\infty}^{\infty} a_j Z_{t-j} + \mu, \quad \sum_j a_j^2 < \infty, \quad \mathbb{E}[Z_0] = 0, \quad \mathbb{E}[Z_0^2] = 1, \tag{6}$$

one can hope to prove a central limit theorem for $S_n(X, \phi)$ or $\tilde{S}_n(X, \phi)$ by resorting to the so-called Bartlett's decomposition. This decomposition amounts to writing

$$I_{n,k}^X = (2\pi) f_X(y_k) \bar{I}_{n,k}^Z + R_{n,k}, \tag{7}$$

where f_X is the spectral density of the process X and $R_{n,k}$ is a remainder term. Note that for a linear process, the spectral density writes

$$f_X(x) = (2\pi)^{-1}|A(e^{ix})|^2 = (2\pi)^{-1}|\sum_{k\in\mathbb{Z}} a_k e^{-ikx}|^2.$$

This decomposition of the periodogram was suggested in Bartlett (1955) and later thoroughly investigated by many authors (see, *e.g.* Walker (1965), Chen and Hannan (1980), Brockwell and Davis (1991) . The leading term in the decomposition (7), $2\pi f_X(x) I^Z_{n,k}$, is sometimes referred to as the *pseudo-periodogram.* It should be stressed however that this quantity is purely of theoretical interest and cannot be estimated explicitly. The Bartlett's decomposition suggests to relate the limiting distribution of $S_n(X,\phi)$ to the limiting distribution of $S_n(Z,\phi)$. A central limit theorem will hold for $S_n(X,\phi)$, provided that we show that $S_n(Z,\phi) - S_n(X,\phi) = o_P(1)$ (*i.e.* converges to zero in probability). This can be achieved (with patience and hard work) under reasonable regularity assumptions on the functional ϕ, which are not considerably more stringent than those needed for the derivation of the central limit theorem for $S_n(Z,\phi)$. We now state our assumptions.

(A1) $(\beta_{n,k})_{1\le k\le K}$ is a triangular array of real numbers such that $\sum_{k=1}^K \beta^2_{n,k} = 1$ and

$$\lim_{n\to\infty} \max_{1\le k\le K} |\beta_{n,k}| = 0.$$

(A2) For all $\epsilon > 0$, $\max_{1\le k\le K} |\beta_{n,k}| = O({\mu_n}^{-1/2+\epsilon})$, where $\mu_n := \#\{k : 1 \le k \le K, \beta_{n,k} \neq 0\}$.

Assumption **(A2)** means that $\mu_n(\max_{1\le k\le K} |\beta_{n,k}|)^2$ is bounded by a slowly varying function of μ_n. It holds in particular when

$$\beta_{n,k} := \left(\sum_{k=1}^K g^2(y_k)\right)^{-1/2} g(y_k)$$

under mild technical condition on function g. Note that, for functions of bounded variations on $[-\pi,\pi]$, $\max_{k\in\{1,\cdots,K\}} |\beta_{n,k}| = O(n^{-1/2})$. Assumption **(A2)** may however hold even when g is not of bounded variation.

(A3) There exists a real τ_m such that

$$\lim_{n\to\infty} n^{-1} \sum_{1\le k\neq l\le K} \beta_{n,k}\beta_{n,l} = \tau_m .$$

For any function $\phi \in \mathcal{H}$ and any real u, define

$$\sigma_m^2(\phi, u) = \mathbb{E}[(\phi(u|\boldsymbol{\xi}|^2/2) - \gamma_m(\phi, u))^2], \tag{8}$$

$$C_m(\phi, u) = \mathbb{E}[(\xi_1^2 - 1)\phi(u|\boldsymbol{\xi}|^2/2)],\ C_m(\phi) := C_m(\phi, 1). \tag{9}$$

(A4) There exists a real $\Sigma_m^2(\phi, f)$ such that

$$\lim_{n\to\infty} \sum_{k=1}^{K} \beta_{n,k}^2 \sigma_m^2(\phi, f(y_k)) = \Sigma_m^2(\phi, f).$$

(A5) There exists a real $\tau_m(\phi, f)$ such that

$$\lim_{n\to\infty} n^{-1} \sum_{1\le k\neq l\le K} \beta_{n,k}\beta_{n,l} C_m(\phi, f(y_k)) C_m(\phi, f(y_l)) = \tau_m(\phi, f).$$

To deal with functions ϕ singular at zero (such as $\log(x)$), an additional technical assumption on the probability distribution of Z_0 is required.

(A6) There exists a real $r \ge 1$ such that $\int_{-\infty}^{+\infty} |\mathbb{E}(e^{itZ_0})|^r dt < \infty$.

Assumption **(A6)** ensures that $n^{-1/2}\sum_{t=1}^{n} Z_t$ has a density q_n for all sufficiently large n ($n \ge p$) and that this density converges uniformly to the standardized Gaussian distribution (see, for example, Bhattacharya and Rao (1976) [3], Theorem 19.1,p.189).

(A7) There exists a real $\delta > 3/4$ such that $\sum_{j\in\mathbb{Z}} |j|^{\delta}|a_j| < \infty$ and $A(z) = \sum_{j\in\mathbb{Z}}^{\infty} a_j z^j \neq 0$ for $|z| = 1$.

Under this assumption, the process is short-range dependent and (perhaps non-causally) invertible.

Theorem 1. *Let $(Z_t)_{t\in\mathbb{Z}}$ be a sequence of i.i.d centered random variables with unit variance and with finite moment of order $\mu \ge 4$, and let $\kappa_4 = \mathbb{E}[Z^4] - 3(\mathbb{E}[Z^2])^2$ be the fourth-order cumulant* *of Z_0. Let X be a linear process which admits the linear representation (6) with respect to Z and such that* **(A7)** *holds. Assume either*

- **(S1)** *ϕ is twice differentiable and there exists an integer ν such that*

$$\max_{x \in \mathbb{R}} \frac{|\phi(x)| + |\phi'(x)| + |\phi''(x)|}{1 + |x|^{\nu}} < \infty,$$

  and $\mu \geq 8\nu \vee 4$.

- **(S2)** *Assumption* **(A6)** *holds, ϕ is twice differentiable on $(0, \infty)$ and there exist integers, $\nu_0 \geq 3$, $\nu_1 \geq 4$, $\nu_2 \geq 3$ and $\nu_3 \geq 3$, reals α and β such that $m > \beta$ and*

$$\int_{\mathbb{R}^{2m}} \phi^2(|x|^2/2)(1 + |x|)^{-\nu_0} dx < \infty \tag{10}$$

$$\int_{\mathbb{R}^{4m}} \phi(|x|^2/2)\phi(|y|^2/2)(1 + |x| + |y|)^{-\nu_1} dx\ dy < \infty \tag{11}$$

$$\int_{\mathbb{R}^{2m}} [\phi'(|x|^2/2)]^4 (1 + |x|)^{-\nu_2} dx < \infty, \tag{12}$$

$$\int_{\mathbb{R}^{4m}} |\phi'(|x|^2/2)||x||\phi'(|y|^2/2)||y|(1 + |x| + |y|)^{-\nu_3} dx\ dy < \infty, \tag{13}$$

$$|\phi''(x)| \leq C(x^{\alpha} \mathbf{1}_{\{x \geq 1\}} + x^{-\beta} \mathbf{1}_{\{x \leq 1\}}), \tag{14}$$

  and $\mu \geq \max\{8, 4\alpha, [2(m - \beta)] + 1, \nu_0, \nu_1, \nu_2, \nu_3 + 2\}$.

Let $(\beta_{n,k})_{1 \leq k \leq K}$ be a triangular array satisfying assumptions **(A1)** *and* **(A2)**.

- *If* **(A3)** *holds then $S_n(X, \phi)$ is asymptotically centered Gaussian with variance $\sigma_m^2(\phi) + \tau_m m^2 C_m^2(\phi) \kappa_4$.*
- *If* **(A4)** *and* **(A5)** *hold then $\tilde{S}_n(X, \phi)$ is asymptotically centered Gaussian with variance $\Sigma_m^2(\phi, f) + \tau_m(\phi, f) m^2 \kappa_4$.*

Remarks.

- In the case of Assumption **(S1)**, the assumptions needed to prove a central limit theorem for $S_n(Z, \phi)$ are sufficient to prove that $S_n(Z, \phi) - S_n(X, \phi) = o_P(1)$. In the case of assumption **(S2)**, (10) and (11) alone imply the central limit theorem for $S_n(Z,\phi)$, while (12), (13) and (14) are needed to prove that $S_n(Z,\phi) - S_n(X,\phi) = o_P(1)$.

- We will only prove the theorem for $S_n(X,\phi)$. Setting $\phi_{n,k}(y) = \phi(f(y_k)z)$, we get

$$\tilde{S}_n(X,\phi) = \sum_{k=1}^{K} \beta_{n,k}\{\phi_{n,k}(\bar{I}_{n,k}^X/f(y_k)) - \gamma_m(\phi_{n,k})\}.$$

  Since we consider only short-range dependent processes with spectral density bounded above and away from zero, Assumptions **(S1)** and **(S2)** hold uniformly for the functions $\phi_{n,k}$ for all n and $k \le K$ whenever they hold for ϕ. This would not be the case if long-range dependent processes were considered. In that case, it is well known that $S_n(X,\phi)$ and $\tilde{S}_n(X,\phi)$ may not have the same asymptotic distribution. In the case of a linear functional, *i.e.* $\phi(x) = x$, $\tilde{S}_n(X,\phi)$ may even have a non Gaussian asymptotic distribution with a suitable normalization, while $S_n(X,\phi)$ is asymptotically Gaussian (cf. Fox and Taqqu, 1983, 1987).
- If $\kappa_4 = 0$ then Assumptions **(A3)** and **(A5)** are not necessary.
- If $\mu_n = o(n^{2/3})$ then Assumption **(A2)** is not necessary and Assumptions **(A3)** and **(A5)** hold with $\tau_m = \tau_m(\phi, f) = 0$.
- If $C_m(\phi, u) = 0$ for all $u \in \mathbb{R}^+$, then Assumptions **(A2)**, **(A3)** and **(A5)** are not necessary. Thus the central limit theorem holds under the same assumption on the weights $\beta_{n,k}$ as in the Gaussian case and with the same asymptotic variance.

3. Applications

3.1. **Estimation of functional of the spectral density.** Many problems in time series analysis require the evaluation of a non-linear functional of the spectral density

$$\Lambda(f) = \int_0^{\pi} w(x)G(f(x))dx, \tag{15}$$

either as a goal or as an intermediate step in an inference procedure. The linear case, $G(x) = x$, is well-understood (cf. Brockwell and Davis, 1991). The estimation of non-linear functional has been scarcely considered in the literature, despite the number of potential applications.

In this section, we study a class of estimators based on the pooled periodogram. For any positive integer m, denote $g_m(t) := t^{m-1}e^{-t}/(m-1)!$, $t \geq 0$, the probability density function of a Gamma$(m, 1)$ random variable. For ϕ a real-valued function such that $\int_0^\infty |\phi(xt)| g_m(t)dt < \infty$ for all $x > 0$, define

$$\mathcal{T}_m[\phi](x) := \int_0^\infty \phi(xt) g_m(t) dt$$

In order to define an estimator of $\Lambda(f)$, we need the following assumption.

(NLF) There exists a function ϕ_m such that, for almost every $x > 0$,

$$\mathcal{T}_m[\phi_m](x) = G(x). \tag{16}$$

When $G(u) = u^\alpha$, with $\alpha + m > 0$, **(NLF)** holds with $\phi_m(x) = \Gamma(\alpha + m)\ x^\alpha$. When $G(u) = \log(u)$, **(NLF)** holds with $\phi_m(x) = \log(x) - \psi(m)$, where $\psi(m)$ is the digamma function. Assumption **(NLF)** implies that $\forall x \in [0, \pi], \mathbb{E}[\phi_m((f(x)|\boldsymbol{\xi}|^2/2))] = G(f(x))$. Define

$$\hat{\Lambda}_n = (2\pi/K) \sum_{k=1}^{K} w(y_k) \phi_m(\bar{I}_{n,k}^X) \tag{17}$$

This construction differs from the estimators considered in Taniguchi (1991) or Mokkadem (1996): these authors suggest to plug in Eq.(15), a consistent estimator of the spectral density.

Theorem 2. *Assume that w is a function of bounded variation. Assume in addition that, for some positive integer m, (16) has a solution denoted ϕ_m, satisfying either assumption (S1) or (S2) of Theorem 1. Then $\sqrt{n}(\hat{\Lambda}_n - \Lambda(f))$ converges weakly to a zero-mean Gaussian distribution.*

The expression of the limiting distribution can be deduced from Theorem 1. In general, the limiting variance depends upon the spectral density $f_X(x)$ of the process X.

Functionals of the log-spectral density are of particular interest. In such case, $G(x) = \log(x)$, and $\phi_m(x) = \log(x) - \psi(m)$. The function ϕ_m is singular at $x = 0$, and so we must verify assumption **(S2)** of Theorem 1. This assumption requires that **(A6)** be verified. It is easily checked that (10)-(14) are satisfied by taking $m \geq 5$ and $E|Z|^\mu < \infty$, for $\mu \leq 2m + 1$. The condition on the pooling size m is imposed by (12), which guarantees that $\mathbb{E}(\phi'_m(\bar{I}^Z_{n,k}))^4 = \mathbb{E}(\bar{I}^Z_{n,k})^{-8} < \infty$. The limiting distribution then is

$$\frac{m\psi'(m)}{\pi}\int_0^\pi w^2(y)dy + \frac{\kappa_4}{4\pi}\left(\int_0^\pi w(y)dy\right)^2, \tag{18}$$

where $\psi'(m)$ is the trigamma function. Note that, not surprisingly, the limiting variance in such case does not depend on $f_X(x)$.

A direct application of the above result is the estimation of the innovation variance. Following the suggestion of Davis and Jones (1968) (see also Hannan and Nichols (1977)), an estimate of the innovation variance may be obtained by computing

$$\hat{\sigma}^2_n = \exp((2\pi K)^{-1}\sum_{k=1}^K(\log(\bar{I}^X_{n,k}) - \psi(m))). \tag{19}$$

These estimators are based on the Kolmogorov formula, which relates the innovation variance to the integral of the log-spectral density, $\sigma^2 = \exp(\int_0^{2\pi}\log f(w)dw/2\pi)$. The estimator (19) has been shown to be consistent by Chen and Hannan (1980). Although hinting that the limiting distribution would involve the fourth order cumulant of Z, these authors failed to prove asymptotic normality in the non-Gaussian case.

It follows from the discussion above that $\sqrt{n}(\hat{\sigma}^2_n - \sigma^2)$ is asymptotically zero-mean Gaussian with variance $(2m\psi'(m) + \kappa_4)\sigma^4$, for $m \geq 5$ and $E|Z|^\mu < \infty$ for $\mu \geq 2m+1$. The recurrence relation $\psi'(m+1) = \psi'(m) - m^{-2}$ indicates that the function $m\psi'(m)$ *decreases* in m, taking values 1.289 at $m=2$ and 1.185 at $m=3$, and tending to 1 as $m \to \infty$. There is thus always some advantages to "pool" the periodogram ordinates prior to compute logged variables. The same comments hold for most applications considered in the sequel (see Taniguchi (1991) chapter 1, for related result).

3.2. **Log-Periodogram Regression.** Log-periodogram regression have been considered by several authors as an alternative to maximum likelihood or the method of moments for parameter estimation (cf. Taniguchi, 1987). For $C < \infty$ and $\delta > 0$, define $\mathcal{F}(C, \delta)$ the space of spectral density defined by

$$\mathcal{F}(C,\delta) := \left\{ g; g(x) = \frac{\sigma^2}{2\pi}\left|\sum_{j=0}^{\infty} a_j e^{-ijx}\right|^2, \sum_{j=0}^{\infty}(1+j)|a_j| < C, \text{ and } \left|\sum_{j=0}^{\infty} a_j z^j\right| \geq \delta, \text{ for all } |z| \leq 1 \right\}$$

We propose to fit some parametric family $\mathcal{P} := \{f_\theta, f_\theta \in \mathcal{F}, \theta \in \Theta \subset \mathbb{R}^p\}$ to the true spectral density $f_X(x)(x) = f(x;\theta_0)$, $\theta_0 \in \text{int}(\Theta)$, of the process by solving the following non-linear least-square criterion

$$S_n(\theta) := \sum_{i=1}^{K_n} \left(\log(\bar{I}_{n,k}^X) - \psi(m) - l(y_k;\theta)\right)^2 \tag{20}$$

where $l(y;\theta) := \log f(y;\theta)$. Any vector $\hat{\theta}_n$ in Θ which minimizes the residual sum of squares (Eq. 20) will be called a least-square log-periodogram regression estimate of θ_0. Based on the results derived above, the theory of log-periodogram estimate parallels the theory of non-linear least-squares estimate, as developed in Jennrich (1969) or Wu (1981) . Denote $D_n(\theta,\theta') := \sum_{i=1}^n (l(y_i;\theta) - l(y_i;\theta'))^2$ and

$$l'(y;\theta) := \left(\frac{\partial}{\partial\theta_j} l(y;\theta)\right)_j, \quad l''(y;\theta) := \left(\frac{\partial^2}{\partial\theta_j\partial\theta_k} l(y;\theta)\right)_{j,k}, \quad j,k \in \{1,\cdots,p\}.$$

We will consider the following assumptions

- **(LPR1)** $n^{-1}D_n(\theta,\theta_0)$ converges uniformly for $(\theta,\theta_0) \in \Theta \times \Theta$ to a continuous function $D(\theta,\theta_0)$, and $D(\theta,\theta_0)$ has a unique minimum at $\theta = \theta_0$.
- **(LPR2)** $l'(y;\theta)$ and $l''(y;\theta)$ exist and are bounded for all θ near θ_0; the true parameter θ_0 is in the interior of Θ and

$$I(\theta_0) := \frac{1}{2\pi}\int_{-\pi}^{\pi} l'(y;\theta_0) l'(y;\theta_0)^T dy$$

  is positive definite.

Note that $I(\theta_0)$ is the Fisher information matrix for Gaussian process. Assumption **(LPR1)** is the Jennrich's classical assumption, allowing to show (in the standard non-linear regression model with i.i.d errors with finite second-order moments) that the non-linear least-square estimator $\hat{\theta}_n$ is strongly consistent. It follows from Theorem 1 that the same assumption may be, in our context, used to show that $\hat{\theta}_n$ is weakly consistent (the proof is by a direct adaptation of Theorem 6 in Jennrich (1969); this assumption can be relaxed; see Wu (1981)). Assumption **(LPR2)** is adapted from Jennrich (1969) to prove the asymptotic normality.

Theorem 3. *Assume* **(A6)**, **(A7)** *and ((**LPR1**)-(**LPR2**)). Assume in addition that* $m \geq 5$ *and* $E|Z|^\mu < \infty$, *for* $\mu \geq 2m+1$. *Then,* $\hat{\theta}_n$ *is* $\mathbb{P}_{\theta_0}$*-weakly consistent and* $\sqrt{n}(\hat{\theta}_n - \theta_0)$ *is* $\mathbb{P}_{\theta_0}$*-asymptotically zero-mean Gaussian with covariance matrix*

$$m\psi'(m) I^{-1}(\theta_0) + \kappa_4/4 I(\theta_0)^{-1/2} J(\theta_0) I(\theta_0)^{-1/2},$$

$$J(\theta_0) = (2\pi)^{-2} \left(\int_{-\pi}^{\pi} l'(y;\theta) dy \right) \left(\int_{-\pi}^{\pi} l'(y;\theta) dy \right)^T$$

The proof is a straightforward application of Jennrich (1969), Theorem 7, and is omitted for brevity. Note that when Z_1 is Gaussian, the least-square log-periodogram regression estimator is quasi-efficient, since $m\psi'(m) \to 1$ very quickly with m.

Taniguchi (1987) suggest to use, instead of the pooled periodogram, a sequence of mean-square consistent estimator of the spectral density in the non-linear least square regression equation 20. When Z_1 is Gaussian, this approach is, from the asymptotical point of view, better than the one outlined here, even though the loss in efficiency is a constant factor $m\psi'(m)$ which is very close to one even for moderate values of m.

In practice of course, there is always a substantial ambiguity to select an appropriate way to construct a preliminary estimate of the spectral density. One can for instance let m tend to infinity and set $\hat{f}(y_k) = m^{-1}\bar{I}_{n,k}$. This will yield an asymptotically efficient estimator. In practice, the values of m will always be rather small (such as $m = 3$ or 4) and $m^{-1}\bar{I}_{n,k}$ can hardly be considered as a consistent estimator. The type of asymptotics adopted here (fixed block size) seems more sensible to derive meaningful confidence interval, since those derived from the postulated limiting distribution of the Taniguchi's estimator are likely to be underestimated for moderate sample sizes.

3.3. **Non-parametric Robust spectral estimation.** The robustification of the usual kernel estimator by means of M-estimation in frequency domain was suggested by Von Sachs (1994) and Janas and Von Sachs (1995) . The procedure, motivated by the robust regression technique (see Härdle and Gasser, 1984), consists in solving in s the estimation equation

$$\hat{\Lambda}_{n,x}(s) = K^{-1} \sum_{k=-K}^{K} W_{b_n}(x - y_k)\phi\left(\frac{\bar{I}^X_{n,k}}{ms} - 1\right), \tag{21}$$

where for any function W and a positive real h, $W_h(x) = h^{-1}W(h^{-1}x)$. To study this estimator, the following assumptions are introduced.

- **(VS1)** W is a Lipschitz symmetric probability density function with compact support included in $[-\pi, \pi]$.
- **(VS2)** ϕ is three times continuously differentiable on $]-1, \infty[$, monotone increasing and ,

$$\int_0^\infty \phi(\frac{x}{m} - 1)g_m(x)dx = 0,$$
$$\max_{x \in \mathbb{R}} \frac{|\phi(x)| + |\phi'(x)| + |\phi''(x)| + |\phi'''(x)|}{1 + |x|^\nu} < \infty.$$

- **(VS3)** $(b_n)_{n \geq 0}$ is a decreasing sequence of positive numbers such that $\lim_{n \to \infty} nb_n^3 = \infty$ and $\lim_{n \to \infty} nb_n^5 = 0$.

To keep the derivations simple, these assumptions are stronger than really needed (in particular, the strict monotonicity can be replaced by a weaker form of local monotonicity; see Von Sachs, 1994). The choice $\phi(x) = x$ leads to the classical (non-robust) "kernel" estimator. The implicit definition of $f_n(x)$ as the root of $\hat{\Lambda}_{n,x}(f_n(x)) = 0$, can be considered as a variant of an M-estimation procedure. It is shown in Von Sachs (1994) and Janas and Von Sachs (1995) that the robustification improves the estimate of the spectral density, in presence of contamination by periodic components (which can be seen as outliers in the frequency domain). The consistency of this estimator is established for linear processes with exponentially decaying filter coefficients ($a_j = O(\rho^{|j|})$, $\rho < 1$) in Janas and Von Sachs (1995). The asymptotic normality of the estimator is obtained only for Gaussian processes (Theorem 3.2, Von Sachs 1994). We will briefly sketch how to exploit Theorem 1 to extend those results to non-necessarily Gaussian linear processes.

Define

$$\sigma_m^2 = \int_0^\infty \phi^2(\frac{x}{m} - 1) g_m(x) dx,$$
$$\beta_m = \int_0^\infty \frac{x}{m} \phi'(\frac{x}{m} - 1) g_m(x) dx,$$
$$\sigma_W^2 = (2\pi)^{-1} \int_{-\pi}^{\pi} W^2(x) dx.$$

Proposition 1. *Assume **(VS1)-(VS3)**. Assume in addition that*

$$\sum_j j^2 |a_j| < \infty, \quad |A(e^{i\omega})| \neq 0, for\ \omega \in [-\pi, \pi]$$

and $E|Z|^\mu < \infty$, *with* $\mu = 8\nu \vee 8$. *Then,*

$$\sqrt{K b_n} \left(\hat{f}_n(x) - f_X(x) \right) \underset{n \to \infty}{\overset{\mathcal{L}}{\Longrightarrow}} \mathcal{N}(0, \sigma_W^2 \sigma_m^2 \beta_m^{-2} f_X^2(x)).$$

An outline of the proof is given in appendix D. The rate of asymptotic normality is the same as in the robust case for the usual kernel estimator. Only the constant in the asymptotic variance will be increased by a factor $\beta_m / \sigma_m^2 \geq 1$, accounting for the asymptotic loss of efficiency due to robustification. Along the same lines, one may generalize the results of Von Sachs (1994) on the bias and the variance of the estimator.

3.4. **Local Polynomial Approximation.** The decomposition $\bar{I}_{n,k}^X = f(y_k) \bar{I}_{n,k}^Z + \bar{R}_{n,k}$ suggests to smooth the periodogram using a least-square method (see Fan and Gijbels, 1996). The log-pooled-periodogram admits the following regression model

$$Y_{n,k}^X := \log(\bar{I}_{n,k}^X) - \psi(m) = l(y_k) + \eta_{n,k} + r_{n,k}$$

where $l(y_k) := \log(f(y_k))$ and $\eta_{n,k} := \log(\bar{I}_{n,k}^Z) - \psi(m)$ and $r_{n,k} := \log(1 + \bar{R}_{n,k} / f(y_k) \bar{I}_{n,k}^Z)$ is an asymptotically negligible remainder term. In the local polynomial regression approach, the unknown regression function l is locally approximated by a polynomial of order p (assuming that the regression function is $(p+1)$-th times differntiable at x). We consider for simplicity a linear fit. For a given x, approximate $l(y)$ locally by $l(y) = a + b(y - x) + O((y - x)^2)$. The local regression estimator may be expressed as

$$\hat{l}_n(x) = \sum_{k=1}^{K_n} \tilde{W}_n((x-y_k)/b_n)Y_{n,k},$$

$$K_n(y) = \frac{1}{b_n}\frac{S_{n,2} - b_n y S_{n,1}}{S_{n,0}S_{n,2} - S_{n,1}^2} W(y)$$

with $S_{n,j} := \sum_{i=1}^{K_n} W_{b_n}(y_k - x)(y_k - x)^j$, where W_{b_n} is defined as in the previous section. This estimator has been suggested (with $m = 1$, *i.e.* without pooling) in an unpublished report by Fan and Kreutzberger (1995) (see also Fan and Gijbels, 1996, section 6.4.1). It is assumed in the sequel that

(LPA1) $\sum_j j^2|a_j| < \infty$ and $|A(e^{ix})| \neq 0$, for $x \in [-\pi, \pi]$,

(LPA2) W is a twice continuously differentiable symmetric probability density function and has a compact support,

(LPA3) $nb_n \to \infty$ and $b_n \to 0$.

Using the results derived above, we may now generalize Theorem 6.4 in Fan and Gijbels (1996).

Proposition 2. *Assume* **(A6)**, ***(LPA1-LPA3)*** *and* $E|Z|^\mu < \infty$, *for* $\mu \geq 2m+1$ *and* $m \geq 5$. *Let* b_n *be a sequence such that* $\lim_{n\to\infty} nb_n^5 = 0$. *Then, for each* x, $0 < x < \pi$,

$$\sqrt{nb_n}\left(\hat{l}_n(x) - l(x)\right) \to_{\mathcal{L}} \mathcal{N}(0, m\psi'(m)\nu_0(W)\pi) \tag{22}$$

where $\mu_j = \int_{-\infty}^{\infty} t^j W(t)dt$,

$$\mu_2(W) = \frac{\mu_2^2 - \mu_1\mu_3}{\mu_0\mu_2 - \mu_1^2},\ \nu_0(W) = \frac{\int_{-\infty}^{\infty}(\mu_2 - \mu_1 t)^2 W^2(t)dt}{(\mu_0\mu_2 - \mu_1^2)^2}.$$

Recall that ψ is the digamma function and note that $\psi'(1) = \pi^2/6$ and (22) in such case corresponds exactly to the statement of Theorem 6.4 in Fan and Gijbels (1996) (formulated by these authors only in the Gaussian case). Since $m\psi'(m) \to 1$, for m large enough, local polynomial regressor estimator is asymptotically equivalent to the local likelihood smoothed periodogram estimator, advocated by Fan and Kreutzberger (1995) (see section 6.4.2 in Fan and Gijbels (1996)), the loss in efficiency being equal to $m\psi'(m)$. Once again, the pooling has a positive effect on the asymptotic distribution of the estimator.

4. Proof of Theorem 1

As mentioned after Theorem 1, we only prove the asymptotic normality of $S_n(X,\phi)$, which will follow from the proof of the asymptotic normality of $S_n(Z,\phi)$ and from the proof that $T_n := S_n(X,\phi) - S_n(Z,\phi) = o_P(1)$. The asymptotic normality is a consequence of Theorem 2 in Fay and Soulier (1999). We now prove that $T_n = o_P(1)$. Denote $T_{n,k} = \phi(\bar{I}^X_{n,k}/f(y_k)) - \phi(2\pi \bar{I}^Z_{n,k})$. A tractable expression for $T_{n,k}$ is naturally obtained by a second order Taylor expansion :

$$\phi(\bar{I}^X_{n,k}/f_X(y_k)) = \phi(2\pi \bar{I}^Z_{n,k}) + \phi'(2\pi \bar{I}^Z_{n,k})\bar{R}_{n,k} + \frac{1}{2}\phi''(2\pi \bar{I}^Z_{n,k} + \theta_{n,k}\bar{R}_{n,k})\bar{R}^2_{n,k}$$

where $0 \le \theta_{n,k} \le 1$ and

$$\bar{R}_{n,k} = \frac{\bar{I}^X_{n,k}}{f_X(y_k)} - 2\pi \bar{I}^Z_{n,k}. \tag{23}$$

Thus we only need to prove the following two lemmas.

Lemma 1. *Assume* **(A1)**, **(A7)** *and either* **(S1)** *or* **(A6)**-*(12)-(13) and* $\mathbb{E}[|Z_t|^\mu] < \infty$ *with* $\mu \ge \nu_2 \vee (\nu_3 + 2) \vee 8$. *Then,*

$$\mathbb{E}\left|\sum_{k=1}^{K} \beta_{n,k}\phi'(2\pi \bar{I}^Z_{n,k})\bar{R}_{n,k}\right| = o(1). \tag{24}$$

Lemma 2. *Assume* **(A1)**,**(A7)** *and either* **(S1)** *or* **(A6)**, *(14) with* $m > \beta$ *and* $\mathbb{E}[|Z_t|^\mu] < \infty$ *with* $\mu \ge 6 \vee 4\alpha \vee ([2m - 2\beta] + 1)$. *Then,*

$$\sum_{k=1}^{K} \beta_{n,k}\phi''\left(2\pi \bar{I}^Z_{n,k} + \theta_{n,k}\bar{R}_{n,k}\right)\bar{R}^2_{n,k} = o_P(1). \tag{25}$$

An important tool in proving these lemmas is the following bound for $\bar{R}_{n,k}$, proved in appendix B.

Lemma 3. *Under Assumption* **(A7)**, *for any integer* p, *if* $\mathbb{E}[Z_0^{2p}] < \infty$,

$$\max_{1\le k\le K} \mathbb{E}[|\bar{R}_{n,k}|^p] = O(n^{-p/2}). \tag{26}$$

Unfortunately, this bound cannot be improved, even by strengthening assumption **(A7)**, and is not sufficient to prove Lemma 1, even if for instance ϕ' is bounded. In this case, we have

$$\mathbb{E}\Big|\sum_{k=1}^{K}\beta_{n,k}\phi'(2\pi\bar{I}^Z_{n,k})R_{n,k}\Big| \le \sum_{k=1}^{K}|\beta_{n,k}|\mathbb{E}[|R_{n,k}|] = O(n^{-1/2})\sum_{k=1}^{K}|\beta_{n,k}| = O(1),$$

since $\sum_{k=1}^K \beta^2_{n,k} = 1$ implies that $\sum_{k=1}^K |\beta_{n,k}| = O(n^{1/2})$. This latter bound cannot generally be improved as shown by the trivial example $\beta_{n,k} = K^{-1/2}$ for all $1 \le k \le K$. The proof of Lemma 1 is very involved and is postponed to Appendix C.

The proof of Lemma 2 is much simpler. Under assumption **(S1)**, it is straightforward if $\nu = 0$. If $\nu > 0$, applying Lemma 3 and Hölder inequality, we get

$$\mathbb{E}[|\sum_{k=1}^{K}\beta_{n,k}\phi''(2\pi\bar{I}^Z_{n,k}+\theta_{n,k}\bar{R}_{n,k})\bar{R}^2_{n,k}|] \le$$
$$\sqrt{K}\max_{1\le k\le K}\Big(\mathbb{E}^{1/2}[\ (2\pi\bar{I}^Z_{n,k})^{2\nu}]\mathbb{E}^{1/2}[\bar{R}^4_{n,k}] + \mathbb{E}[|\bar{R}_{n,k}|^{\nu+2}]\Big) = O(n^{-1}),$$

provided Z_0 has finite moments up to the order $4\nu \vee 8$.

Under assumption **(S2)**, ϕ might be singular at zero, and this induces some more technicalities and necessitates Assumption **(A6)**. Under assumptions **(A7)** and **(A6)**, for all $n \ge 2r$ (where r is defined in assumption **(A6)**), it holds that $\mathbb{P}(\bar{I}^Z_{n,k} > 0) = \mathbb{P}(\bar{I}^X_{n,k} > 0) = 1$. Thus $\bar{U}_{n,k} = \bar{R}_{n,k}/(2\pi\bar{I}^Z_{n,k})$ is well defined and we have the following lemma, proved in appendix B.

Lemma 4. *Assume* **(A7)**, **(A6)**, $\mathbb{E}[Z_0^6] < \infty$ *and* $m \ge 1$. *Then,*

$$\max_{1\le k\le K}|\bar{U}_{n,k}| = o_P(1).$$

Denote $\Omega_n = \{\max_{1\le k\le K}|\bar{U}_{n,k}| \le 1/2\}$. Lemma 4 implies that $\mathbb{P}(\Omega_n) = 1$, thus it is sufficient to prove Lemma 1 on Ω_n. On that event, it holds that

$$2\pi\bar{I}^Z_{n,k}/2 \le 2\pi\bar{I}^Z_{n,k} + \theta_{n,k}\bar{R}_{n,k} \le 6\pi\bar{I}^Z_{n,k}/2.$$

Applying (14), we get

$$|\phi''(2\pi\bar{I}^Z_{n,k}+\theta_{n,k}\bar{R}_{n,k})|\mathbf{1}_{\Omega_n}\leq C(|\bar{I}^Z_{n,k}|^\alpha+|\bar{I}^Z_{n,k}|^{-\beta}),$$
$$\big|\mathbb{E}[\sum_{k=1}^K\beta_{n,k}\phi''(2\pi\bar{I}^Z_{n,k}+\theta_{n,k}\bar{R}_{n,k})\bar{R}^2_{n,k}\mathbf{1}_{\Omega_n}]\big|\leq C\sqrt{n}\max_{1\leq k\leq K}\mathbb{E}^{1/2}[\bar{R}^4_{n,k}]\mathbb{E}^{1/2}[(\bar{I}^Z_{n,k})^{2\alpha}+(\bar{I}^Z_{n,k})^{-2\beta}].$$

Provided $\mathbb{E}[|Z_0|^{4\alpha}<\infty$, $\mathbb{E}[(\bar{I}^Z_{n,k})^{2\alpha}]$ is uniformly bounded. To conclude, we must prove that $\mathbb{E}[(\bar{I}^Z_{n,k})^{-2\beta}]$ is also uniformly bounded. The following Lemma is a consequence of Lemma 7 in appendix A.

Lemma 5. *Let $\gamma>0$ and let m and μ be two integers such that $m>\gamma$ and $\mu>2m-2\gamma$. Under Assumption* **(A6)**, *if $\mathbb{E}[|Z|^\mu]<\infty$, then there exists an integer $n_0\geq 2m$, which depends only on the distribution of Z_0 such that*

$$\max_{n\geq n_0}\max_{1\leq k\leq K}\mathbb{E}[(\bar{I}^Z_{n,k})^{-\gamma}]<\infty$$

Applying Lemmas 3 and 5 yields

$$\mathbb{E}[\sum_{k=1}^K\beta_{n,k}\phi''(2\pi\bar{I}^Z_{n,k}+\theta_{n,k}\bar{R}_{n,k})\bar{R}^2_{n,k}\mathbf{1}_{\Omega_n}]=O(n^{-1/2}).$$

This concludes the proof of Lemma 2 and Theorem 1.

Appendix A. Edgeworth expansions

The Lemmas needed to prove Theorem 1 make use of the technique of Edgeworth expansions. In this section, we briefly sketched the main notations and results that are required in the sequel. We follow Battacharya and Rao (1976) and Götze and Hipp (1978).

Define, for any q-tuple $\mathbf{k}=(k(1),\cdots,k(q))$, $k(i)\in\{1,\cdots,K\}$, $k(i)\neq k(j)$ for all $i,j\in\{1,\cdots,q\}$,

$$\mathbf{W}_n(\mathbf{k})=(2/n)^{1/2}\sum_{j=1}^n Z_j[\cos(x_{k(1)}j),\sin(x_{k(1)}j),\cdots,\cos(x_{k(q)}j),\sin(x_{k(q)}j)]^T. \quad (27)$$

Let κ_r denote the cumulant of order r of Z_1. For any $2q$-uplet of non-negative integers $\boldsymbol{\nu} := (\nu(1), \nu(2), \cdots, \nu(2q))$, let $|\boldsymbol{\nu}| := \sum_{i=1}^{2q} \nu(i)$ and $\boldsymbol{\nu}! = \nu(1)! \cdots \nu(2q)!$. Define the average cumulant of order $\boldsymbol{\nu}$ as

$$\chi_{n,\boldsymbol{\nu}}(\mathbf{k}) := 2^{|\boldsymbol{\nu}|/2} n^{-1} \mathrm{cum}(\underbrace{\sum_{t=1}^{n} Z_t \cos(x_{k(1)}t), \cdots}_{\nu(1)\text{times}}, \underbrace{\sum_{t=1}^{n} Z_t \sin(x_{k(q)}t) \text{ times}}_{\nu(2q)}),$$

$$= 2^{|\boldsymbol{\nu}|/2} \kappa_{|\boldsymbol{\nu}|} n^{-1} \sum_{t=1}^{n} \cos^{\nu(1)}(x_{k(1)}t) \sin^{\nu(2)}(x_{k(1)}t) \cdots \sin^{\nu(2q-1)}(x_{k(q)}t) \cos^{\nu(2q)}(x_{k(q)}t).$$

It is important to note that $\chi_{n,\boldsymbol{\nu}}(\mathbf{k})$ is bounded uniformly with respect to $\mathbf{k}$. For all $\mathbf{k}$, the latter equation implies that $|\chi_{n,\boldsymbol{\nu}}(\mathbf{k})| \leq 2^{\boldsymbol{\nu}/2} \kappa_{|} \boldsymbol{\nu}|$. Denote $\chi_n(\mathbf{k}) := (\chi_{n,\boldsymbol{\nu}}(\mathbf{k}))_{\boldsymbol{\nu} \in \mathbb{N}^{2q}}$ and

$$P_r(\mathbf{x}, \chi_n(\mathbf{k})) = \sum_{u=1}^{r} \frac{1}{u!} \sum\nolimits_{u,r}^{*} \frac{\chi_{n,\boldsymbol{\nu}_1}(\mathbf{k}) \cdots \chi_{n,\boldsymbol{\nu}_u}(\mathbf{k})}{\boldsymbol{\nu}_1! \cdots \boldsymbol{\nu}_u!} H_{\boldsymbol{\nu}_1 + \cdots + \boldsymbol{\nu}_u}(\mathbf{x}) \tag{28}$$

where $\sum_{u,r}^{*}$ extends over all u-uplets $(\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_u)$ of $(2q)$-uplets $\boldsymbol{\nu}_i$, $1 \leq i \leq u$ verifying the two properties

$$3 \leq |\boldsymbol{\nu}_i| \text{ and } \sum_{k=1}^{u} (|\boldsymbol{\nu}_k| - 2) = r.$$

and for any $2q$-uplets of integers $\boldsymbol{\nu} = (\nu(1), \cdots, \nu(2q))$, $H_{\boldsymbol{\nu}}$ is the multi-dimensional Hermite polynomial of order $\boldsymbol{\nu}$, defined as

$$H_{\boldsymbol{\nu}}(\mathbf{x}) = H_{\nu(1)}(x_1) \cdots H_{\nu(2q)}(x_{2q}),$$

H_i denoting the i-th scalar Hermite polynomial.

Let d be a positive integer, and let $(k_1, \cdots, k_d)$ be a d-tuple of pairwise distinct integers ($k_i \in \{1, \cdots, K\}$, $k_i \neq k_j$, for all $i, j \in \{1, \cdots, d\}$). Denote

$$\mathbf{k} := [m(k_1 - 1) + 1, \cdots, mk_1, \cdots, m(k_d - 1) + 1, \cdots, mk_d]. \tag{29}$$

Let $\boldsymbol{\xi}$ denote a standard $2dm$-dimensional Gaussian vector. For a function $\phi : \mathbb{R}^{2dm} \to \mathbb{C}$ such that $\mathbb{E}[\phi^2(\boldsymbol{\xi})] < \infty$, define, for $r, u \in \mathbb{N}$,

$$\mathbb{E}_{r,\mathbf{k}}(\phi) = \mathbb{E}[P_r(\boldsymbol{\xi}, \chi_n(\mathbf{k}))\phi(\boldsymbol{\xi})], \tag{30}$$

$$Q_{u,\mathbf{k}}(\phi) = \sum_{r=0}^{u} n^{-r/2}\mathbb{E}_{r,\mathbf{k}}(\phi). \tag{31}$$

$Q_{u,\mathbf{k}}(\phi)$ is the u-th order Edgeworth expansion of the moment $\mathbb{E}[\mathbf{W}_n(\mathbf{k})]$. Since $\chi_n(\mathbf{k})$ is bounded with respect to $\mathbf{k}$, we also have the following uniform bound for $\mathbb{E}_{r,\mathbf{k}}$.

$$|\mathbb{E}_{r,\mathbf{k}}(\phi)| \le C_r \mathbb{E}^{1/2}[\phi^2(\boldsymbol{\xi})]. \tag{32}$$

In order to obtain conditions upon which the error $\mathbb{E}[\phi(\mathbf{W}_n(\mathbf{k})] - Q_{u,\mathbf{k}}(\phi)$ is controlled, we need to define some functional spaces. Denote $C^q(\mathbb{R}^a)$ the set of all functions on $\mathbb{R}^a$ with continuous derivative of order q. For an a-tuple of integers $\boldsymbol{\beta} = (\beta(1), \cdots, \beta(a))$, denote $D^{\boldsymbol{\beta}} = \partial_{x_1}^{\beta(1)} \cdots \partial_{x_a}^{\beta(a)}$. For $\nu \in \mathbb{N}$, and any measurable function ψ on $\mathbb{R}^a$, define

$$N_{a,\nu}(\psi) = \int_{\mathbb{R}^a} |\psi(x)|(1 + |x|)^{-\alpha} dx,$$

$$M_{a,\nu}(\psi) = \sup_{x \in \mathbb{R}^a} \frac{|\psi(x)|}{1 + |x|^{\nu}}.$$

Finally, for $\nu, r \in \mathbb{N}$, let $\mathcal{S}_\nu^r(\mathbb{R}^a)$ be the subspace of $C^r(\mathbb{R}^a)$ such that for all d-tuples of integers that satisfy $\beta(1) + \cdots + \beta(a) \le r$, $M_{a,\nu}(D^{\boldsymbol{\beta}}\psi) < \infty$, for $\psi \in \mathcal{S}_\nu^r(\mathbb{R}^a)$, define

$$\tilde{M}_{a,\nu,r}(\psi) = \max_{\beta(1)+\cdots+\beta(a) \le r} M_{a,\nu}(D^{\boldsymbol{\beta}}\psi).$$

Lemma 6. ***(BR)*** *If* **(A6)** *holds and if there exists $u \in \mathbb{N}^*$ such that $N_{2md,u+2}(\phi) < \infty$ and $\mathbb{E}[|Z_0|^{u+2}] < \infty$, then*

$$\max_{1 \le k \le K} |\mathbb{E}[\phi(\mathbf{W}_n(\mathbf{k})] - Q_{u,\mathrm{k}}(\phi)| \le n^{-u/2}\epsilon_n N_{2md,u+2}(\phi), \tag{33}$$

(GH) *If there exist positive integers u, p such that $\mathbb{E}[|Z_0|^{u+2}] < \infty$, $\phi \in \mathcal{S}_\nu^r(\mathbb{R}^{2dm})$, then*

$$\max_{1 \le k \le K} |\mathbb{E}[\phi(\mathbf{W}_n(\mathbf{k})] - Q_{u,\mathrm{k}}(\phi)| \le n^{-u/2}\epsilon_n \tilde{M}_{2md,\nu,r}(\phi) + C\tilde{M}_{2md,\nu,r}(\phi) n^{-(r+2dm+1)/2}, \tag{34}$$

where, in both cases, $(\epsilon_n)_{n\in\mathbb{N}}$ is a sequence (depending only on the distribution of Z_0, d and m) such that $\lim_{n\to\infty} \epsilon_n = 0$.

Remarks.

- **(BR)** is a consequence of Theorem 19.7 of Battacharya and Rao (1976). **(GH)** is implied by Theorem 3.17 of Gotze and Hipp (1978). In that case, **(A6)** is not required, thus the result applies even when the probability distribution of Z_0 has a discrete support.
- As noted by Chen and Hannan (1980), Theorem 19.7 of Bhattacharya and Rao (1976) applies for normalized sums of i.i.d sequences, but not for triangular arrays. However, they proved for their Lemma 1 that the conclusion of Theorem 19.7 in Bhattacharya and Rao (1976) still holds in our case, upon replacing the cumulants by the averaged cumulants.

We now derive some consequences of Lemma 6. In the case of a Gaussian white noise, since the sequence $2\pi\bar{I}^Z_{n,k}$, $1 \le k \le K$ is i.i.d. $\Gamma(m,1)$, a positive fonction ϕ satisfies either $\mathbb{E}[\phi(2\pi\bar{I}^Z_{n,k})] = \infty$ or $\mathbb{E}[\phi(2\pi\bar{I}^Z_{n,k})]$ is uniformly bounded with respect to n and k. In the non Gaussian case, if ϕ is polynomially bounded, then $\mathbb{E}[\phi(2\pi\bar{I}^Z_{n,k})]$ is uniformly bounded with respect to n and k under a relevant moment condition on Z_0. If ϕ has singularity, then nothing can be said in full generality, and for a given n and k, $\mathbb{E}[\phi(2\pi\bar{I}^Z_{n,k})]$ may not even exist. Under assumption **(A6)**, and under an integrability condition such as (10), $\mathbb{E}[\phi(2\pi\bar{I}^Z_{n,k})]$ can be proved to be finite and uniformly bounded, for large enough n.

Lemma 7. *Let $m \ge 1$ and $s \ge 3$ be integers and let ϕ be a function such that (10) holds* i.e. $\int_{\mathbb{R}^{2m}} |\phi(|x|^2)|(1+|x|)^{-s}dx < \infty$. *Under assumption* **(A6)**, *if $\mathbb{E}[|Z_0|^s] < \infty$, then there exists an integer $n_0 \ge 2m$ which depends only on the distribution of Z_0, m and s such that for all $n \ge n_0$,*

$$\max_{1\le k\le K} \mathbb{E}[|\phi(\bar{I}^Z_{n,k})|] \le C\big(\mathbb{E}^{1/2}[\phi^2(|\boldsymbol{\xi}|^2/2)] + N_{2m,s}(\phi)\big), \tag{35}$$

$$\max_{1\le k\le K} \big|\mathbb{E}[\phi(\bar{I}^Z_{n,k})] - \mathbb{E}[\phi(\boldsymbol{\xi})]\big| \le O(n^{-1})\mathbb{E}^{1/2}[\phi^2(|\boldsymbol{\xi}|^2/2)] + o(n^{(s-2)/2})N_{2m,s}(\phi), \tag{36}$$

where the constant C and the terms $O(n^{-1})$ and $o(n^{(s-2)/2})$ depend only on the distribution of Z_0, m and s but not on ϕ.

Lemma 7 is a straightforward consequence of Lemma 6 **(BR)** and of the bound (32). Lemma 7 implies Lemma 5 in section 4 and the following Lemma which is used in the proof of Lemma 4 in section B.

Lemma 8. *Assume* **(A6)** *and* $\mathbb{E}[Z_0^6] < \infty$. *Then, for all* $\epsilon \in [0,1]$,

$$\max_{1 \le j \le K} \mathbb{P}(\bar{I}^Z_{n,j} \le \epsilon) \le \frac{\epsilon^m}{m!} + O(n^{-2}),$$

where the term $O(n^{-1})$ *depends only on* m *and on the distribution of* Z_0, *and is uniform with respect to* ϵ *in* $[0,1]$.

Lemma 9. *Let* p, q, s *be positive integers with* $s \ge 3$ *and let* $T : \mathbb{R}^{2q} \to \mathbb{R}$ *be a measurable function. Assume that* $\mathbb{E}[|Z_0|^{s+p}] < \infty$ *and that one of the following assumptions hold.*

- ***(BR)*** **(A6)** *holds and* $N_{2q,s}(T) < \infty$.
- ***(GH)*** *There exists an integer* $r \ge 2$ *such that* $T \in \mathcal{S}^r_s(\mathbb{R}^{2q})$.

Then, $\max_{\mathbf{k}} \max_{\mathbf{t}} |\mathbb{E}[T(W_n(\mathbf{k}))\mathbf{Z}_{\mathbf{t}}] - \mathbb{E}[T(\boldsymbol{\xi})]\ \mathbb{E}[\mathbf{Z}_{\mathbf{t}}]| = O(n^{-1/2})$,

where $\max_{\mathbf{k}}$ *is the maximum over all the* q*-uplets* $\mathbf{k} = (k_1, \dots, k_q)$ *of pairwise distinct integers in* $\{1, \dots, K\}$, $\max_{\mathbf{t}}$ *is the maximum over all* p*-uplets of (not necessarily distinct) integers* $\mathbf{t} = (t_1, \cdots, t_p)$, $1 \le t_i \le n$ *and* $\mathbf{Z}_{\mathbf{t}} := Z_{t_1} \cdots Z_{t_p}$ *and, as usual,* $\boldsymbol{\xi}$ *denotes a* $2q$ *dimensional standard Gaussian vector.*

Proof of Lemma 9. Write $W_n(\mathbf{k}) = \tilde{W}_n(\mathbf{k}) + Y_n(\mathbf{k})$ with

$$\tilde{W}_n(\mathbf{k}) = (2\pi n)^{-1/2} \sum_{t \in \{1,\cdots,K\} \setminus \{t_1,\cdots,t_p\}} Z_t \left[\cos(x_{k_1} t), \sin(x_{k_1} t), \dots, \cos(x_{k_q} t), \sin(x_q t)\right]^T$$

$$Y_n(\mathbf{k}) = (2\pi n)^{-1/2} \sum_{t \in \{t_1,\cdots,t_p\}} Z_t \left[\cos(x_{k_1} t), \sin(x_{k_1} t), \dots, \cos(x_{k_q} t), \sin(x_{k_q} t)\right]^T.$$

Define $\psi(x) = \mathbb{E}[T(x + \boldsymbol{\xi})]$ and $G_n(x) = \mathbb{E}[T(x + \tilde{W}_n(\mathbf{k}))] - \psi(x)$. Under the assumptions of Lemma 9, ψ and G_n are well defined and $\mathbb{E}[|\psi(Y_n(\mathbf{k}))\mathbf{Z}_{\mathbf{t}}|] < \infty$ and $\mathbb{E}[|G_n(Y_n(\mathbf{k}))\mathbf{Z}_{\mathbf{t}}|] < \infty$.

Moreover, with these notations, $\mathbb{E}[T(\boldsymbol{\xi})]\ \mathbb{E}[\mathbf{Z_t}] = \psi(0)\mathbb{E}[\mathbf{Z_t}]$. Thus, we get

$$\begin{aligned}\mathbb{E}[T(W_n(\mathbf{k})\mathbf{Z_t}] &= \mathbb{E}[\mathbb{E}[T(W_n(\mathbf{k}))|\mathbf{Z_t}]\mathbf{Z_t}] = \mathbb{E}[\psi(Y_n(\mathbf{k}))\mathbf{Z_t}] + \mathbb{E}[G_n(Y_n(\mathbf{k}))\mathbf{Z_t}] \\ &= \psi(0)\ \mathbb{E}[\mathbf{Z_t}] + \mathbb{E}[\{\psi(Y_n(\mathbf{k})) - \psi(0)\}\mathbf{Z_t}] + \mathbb{E}[G_n(Y_n(\mathbf{k}))\mathbf{Z_t}].\end{aligned}$$

The assumptions of Lemma 9 also imply that ψ is continuously differentiable and there exists a constant $C(T)$ such that $|\psi(x)|+|\psi'(x)| \le C(T)(1+|x|)^s$. In order to avoid assuming unnecessary moment condition on Z_0, we use the following trick. Note that for all u,

$$\begin{aligned}|\psi(u) - \psi(0)| &= |\psi(u) - \psi(0)|\mathbf{1}_{\{|u|\le 1\}} + |\psi(u) - \psi(0)|\mathbf{1}_{\{|u|\ge 1\}} \\ &\le C(T)(1+|u|)^s|u|\mathbf{1}_{\{|u|\le 1\}} + 2C(T)(1+|u|)^s\mathbf{1}_{\{|u|\ge 1\}} \le 2^{s+1}C(T)(|u| + |u|^s).\end{aligned}$$

Using brutally the mean value theorem yields the bound $C(T)|u|(1+|u|)^s$, which would increase the required number of moments for Z_0. We now trivially have

$$|\mathbb{E}[(\psi(Y_n(\mathbf{k})) - \psi(0))\mathbf{Z_t}]| \le C\mathbb{E}[(|Y_n(\mathbf{k})| + |Y_n(\mathbf{k})|^s)|\mathbf{Z_t}|] = O(n^{-1/2}),$$

provided Z_0 has $s+p$ moments.

To bound $\mathbb{E}[G_n(Y_n(\mathbf{k}))\mathbf{Z_t}]$, we must first check the validity of an Edgeworth expansion of $G_n(x) = \mathbb{E}[T(x + \tilde{W}_n(\mathbf{k}))]$:

$$G_n(x) = \sum_{r=0}^{s-2} n^{-r/2}\tilde{\mathbb{E}}_{r,\mathbf{k}}[T(x+.)] + o(n^{-(s-2)/2})R_{s,T}(x).$$

Two problem arise. The first problem is that the validity of an Edgeworth expansion has been proved only for $W_n(k)$ and not for $\tilde{W}_n(k)$. Nevertheless, the arguments of Chen and Hannan (1980) still apply. It is also necessary to obtain a bound for the terms $\tilde{\mathbb{E}}_{r,\mathbf{k}}$. The difference between $W_n(k)$ and $\tilde{W}_n(k)$ is that the covariance matrix, say $\tilde{\Sigma}_n$, of $\tilde{W}_n(k)$ is not the identity matrix. Instead, it writes $\tilde{\Sigma}_n = J_{2q} + \Delta_n$, where the entries of Δ_n are uniformly $O(n^1)$. Since q is fixed, it is then easily seen that for any function ϕ that satisfies assumption (**BR**) or (**GH**) $\left|\tilde{\mathbb{E}}_{r,\mathbf{k}}[\phi]\right| \le C(\phi)$ where $C(\phi)$ is the relevant norm for ϕ. The second one, which is easily solved, is to give a bound for the remainder term $R_{s,T}(x)$. Under the assumptions of lemma 9, it is easily checked that $|R_{s,T}(x)| \le C(T)(1+|x|)^s$. Thus, we get, noting moreover that by definition of T, $\mathbb{E}_{0,\mathbf{k}}[T(x+.)]=0$,

$$\mathbb{E}[G_n(Y_n(\mathbf{k}))Z_\mathbf{t}]=\sum_{r=1}^{s-2} n^{-r/2}\mathbb{E}[\mathbb{E}_{r,\mathbf{k}}[T(Y_n(\mathbf{k})+.)]\mathbf{Z}_\mathbf{t}]+o(n^{-(s-2)/2})\mathbb{E}[R_{s,T}(Y_n(\mathbf{k})\mathbf{Z}_\mathbf{t}].$$

Under the assumptions of lemma 9, it is also easily checked that for any polynomial $P:\mathbb{R}^{2q}\to\mathbb{R}$, $\mathbb{E}_0[P(\xi)T(x+\xi)]\le C(P)(1+|x|)^s$. Thus

$$\left|\mathbb{E}_{r,\mathbf{k}}[T(Y_n(\mathbf{k})+.)]\mathbf{Z}_\mathbf{t}]\right|\le C\mathbb{E}[(1+|Y_n(\mathbf{k})|)^s\mathbf{Z}_\mathbf{t}]<\infty,$$
$$\left|\mathbb{E}[R_{s,T}(Y_n(\mathbf{k})\mathbf{Z}_\mathbf{t}]\right|\le \mathbb{E}[(1+|Y_n(\mathbf{k})|)^s\mathbf{Z}_\mathbf{t}]<\infty.$$

provided $\mathbb{E}[|Z_0|^{p+s}]<\infty$, and the bounds are uniform with respect to n and $k\le K$. This concludes the proof of Lemma 9.

Appendix B. Bounds related to the Bartlett decomposition

In this section, we bound the error incurred by the approximation of the periodogram of a linear process by the pseudo-periodogram. Most of the material in this section is standard (cf. Walker, 1965).

Denote $r_n(x):=d_n^X(x)-A(e^{ix})d_n^Z(x)$. Following Walker (1965), $r_n(x)$ may be expressed as $r_n(x)=r_n^{(1)}(x)+r_n^{(2)}(x)$, with for $l=1,2$,

$$r_n^{(l)}(x):=(-1)^l(2\pi n)^{-1/2}\sum_{u=-\infty}^{\infty}a_ue^{-iux}Z_{n,u}^{(l)}(x) \tag{37}$$

where

$$Z_{n,u}^{(l)}(x)=\sum_{v\in\mathcal{I}_{n,u}^{(l)}}Z_ve^{-ivx}, \tag{38}$$

$$\mathcal{I}_{n,u}^{(1)}:=\{(1-u),\cdots,(n-u)\}\cap\{1,\cdots,n\}^c. \tag{39}$$

$$\mathcal{I}_{n,u}^{(2)}:=\{1,\cdots,n\}\setminus(\{(1-u),\cdots,(n-u)\}\cap\{1,\cdots,n\}). \tag{40}$$

Note that $\mathcal{I}_{n,u}^{(1)}\subset\{1,\cdots,n\}$ and $\mathcal{I}_{n,u}^{(2)}\subset\{1,\cdots,n\}^c$, and $\#[\mathcal{I}_{n,u}^{(1)}]=\#[\mathcal{I}_{n,u}^{(2)}]=\min(n,|u|)$. The following result is a straightforward adaptation of Theorems 2 and 2a in Walker (1965).

Lemma 10. *Assume* $\sum_j |j|^{1/2}|a_j| < \infty$ *and* $\mathbb{E}|Z|^p < \infty$ *for some* $p \geq 2$. *Then,*

$$\max_{0\leq x\leq 2\pi} \mathbb{E}[|d_n^Z(x)|^p] = O(1), \tag{41}$$

$$\max_{0\leq x\leq 2\pi} \mathbb{E}[|r_n^{(l)}(x)|^p] = O(n^{-p/2}),\ l = 1,2 \tag{42}$$

Assume $\sum_j |j|^{1/4+\epsilon}|a_j| < \infty$, *for some* $\epsilon > 0$. *Then,*

$$\mathbb{E}[\max_{1\leq k\leq n} |r_n^{(l)}(x_k)|] = O(n^{-\epsilon}), \quad l = 1,2. \tag{43}$$

Proof Lemma 10. Eq. (41) is a direct consequence of Rosenthal inequality (see, *e.g.* , Petrov, 1985, Theorem 2.9). This inequality also implies

$$\max_{x\in[0,\pi]} \mathbb{E}|Z_{n,u}^{(l)}(x)|^p \leq c(p)\min(n,|u|)^{p/2}.$$

Minkowski inequality yields, for $l = 1,2$,

$$\max_{x\in[0,\pi]} (\mathbb{E}|r_n^{(l)}(x)|^p)^{1/p} \leq cn^{-1/2}\sum_{u=-\infty}^{\infty} |a_u|\min(n,|u|)^{1/2} = O(n^{-1/2}).$$

which proves (42). Let $u < v$, $u, v \in \mathbb{N}$. We have

$$\Big|\sum_{t=u+1}^{v} Z_t e^{-ikx}\Big|^2 = \sum_{\tau=-(v-u)+1}^{(v-u)-1} e^{-i\tau x}\sum_{t=u}^{v-|\tau|} Z_t Z_{t+|\tau|} \leq \sum_{\tau=-(v-u)+1}^{(v-u)-1}\Big|\sum_{t=u+1}^{v-|\tau|} Z_t Z_{t+|\tau|}\Big|,$$

$$\mathbb{E}\Big[\max_{1\leq k\leq n}\Big|\sum_{t=u+1}^{v} Z_t e^{-ikx}\Big|^2\Big] \leq \mathbb{E}\Big[\sum_{t=u+1}^{v} Z_t^2\Big] + 2\sum_{\tau=1}^{(v-u)-1}\mathbb{E}\Big[\Big(\sum_{t=u}^{v-|\tau|} Z_t Z_{t+|\tau|}\Big)^2\Big]^{1/2} \leq c(v-u)^{3/2}.$$

This yields $\mathbb{E}[\max_{1\leq k\leq n}|\sum_{t=u+1}^{v} Z_t e^{-ikx}|] \leq c(v-u)^{3/4}$ and

$$\mathbb{E}[\max_{1\leq k\leq n} |Z_{n,u}^{(l)}(x_k)] \leq c\min(n,|u|)^{3/4}, \quad l = 1,2.$$

Plugging this expression in (37) finally gives

$$\begin{aligned}\mathbb{E}[\max_{1\leq k\leq n} |r_n^{(l)}(x_k)|] &\leq cn^{-1/2}\sum_{|u|\leq n} |u|^{3/4}|a_u| + n^{1/4}\sum_{|u|>n}|a_u|,\\ &\leq cn^{-\epsilon}\sum_u |u|^{1/4+\epsilon}|a_u|.\end{aligned}$$

which concludes the proof of (43).

Define $R_n(x) := I_n^X(x) - |A(e^{ix})|^2 I_n^Z(x)$ and $\bar{R}_{n,k} := \sum_{l \in J_k} R_n(x_l)/(2\pi f_X(y_k))$. We have

$$R_n(x) = 2\mathrm{Re}\left(A(e^{ix}) d_n^Z(x) r_n(x)\right) + |r_n(x)|^2,$$
$$\bar{R}_{n,k} = \frac{1}{2\pi f_X(y_k)} \sum_{l \in J_k} R_{n,l} + \sum_{l \in J_k} \frac{f_X(x_k) - f_X(y_k)}{f_X(y_k)} I_{n,k}^Z.$$

Decompose $\bar{R}_{n,k} = \sum_{j=1}^4 \bar{R}_{n,k}^{(j)}$, with

(44) $$\bar{R}_{n,k}^{(l)} = 2 \sum_{l \in J_k} \mathrm{Re}\left(d_{n,l}^Z r_{n,l}^{(j)} A(e^{ix_l})\right) / (2\pi f_X(y_k)), l = 1, 2.$$

(45) $$\bar{R}_{n,k}^{(3)} = \sum_{l \in J_k} |r_{n,l}|^2 / (2\pi f_X(y_k)),$$

(46) $$\bar{R}_{n,k}^{(4)} = \sum_{l \in J_k} f_X^{-1}(y_k)(f_X(x_l) - f_X(y_k)) \bar{I}_{n,l}^Z.$$

Lemma 11. *Assume* **(A7)** *and* $\mathbb{E}Z^{2p} < \infty$ *that*

(47) $$\max_{1 \le k \le K} \mathbb{E}[|\bar{R}_{n,k}^{(l)}|^p] = O(n^{-p/2}), \ j = 1, 2,$$

(48) $$\max_{1 \le k \le K} \mathbb{E}[|\bar{R}_{n,k}^{(3)}|^p] = O(n^{-p}),$$

(49) $$\max_{1 \le k \le K} \mathbb{E}[|\bar{R}_{n,k}^{(4)}|^p] = O(n^{-p(\delta \wedge 1)}).$$

The proof of (47) and (48) is a straightforward application of Lemma 10 and do not need assumption **(A7)** in its full strength. To prove, (49), note that, under **(A7)**, it holds that $|f_X(x) - f_X(y)| \le C|x - y|^{\delta \wedge 1}$ for all x, y and thus

(50) $$\max_{1 \le k \le K} \max_{l \in J_k} |f_X(x_l) - f_X(y_k)| / f_X(y_k) = O(n^{-\delta \wedge 1}).$$

Since $\mathbb{E}[(\bar{I}_{n,k}^Z)^p]$ is uniformly bounded provided $\mathbb{E}[Z_0^{2p}] < \infty$, (49) holds.

Proof of Lemma 4. Write $U_{n,k} = U_{n,k}^{(1)} + U_{n,k}^{(2)}$ with

$$U_{n,k}^{(1)} = \frac{\sum_{l \in J_k} 2\mathrm{Re}\left(d_n^Z(x_l) A(e^{ix_l}) r_n(x_l)\right) + |r_n(x_l)|^2}{(2\pi) f_X(y_k) \bar{I}_{n,k}^Z}$$
$$U_{n,k}^{(2)} = \frac{1}{f_X(y_k)} \sum_{l \in J_k} \frac{I_{n,l}^Z}{\bar{I}_{n,k}^Z} (f_X(x_l) - f_X(y_k))$$

Under **(A7)**, $A(e^{ix})$ (and $f_X(x)$) are bounded and bounded away from zero. Applying Cauchy-Schwarz inequality, we get,

$$\left|U_{n,k}^{(1)}\right| \le C(\gamma_{n,k} + \gamma_{n,k}^2), \text{with} \quad \gamma_{n,k} = \left(\bar{I}_{n,k}^Z\right)^{-1/2} \max_{1 \le l \le n} |r_n(x_l)|. \tag{51}$$

Let $(M_n)_{n \in \mathbb{N}}$ be an arbitrary sequence of positive numbers. We have

$$\begin{aligned}\mathbb{P}(\max_{1 \le k \le K} |\gamma_{n,k}| \ge \epsilon) &\le \mathbb{P}\left(\min_{1 \le k \le K} \bar{I}_{n,k}^Z \le \frac{M_n^2}{\epsilon^2}\right) + \mathbb{P}(\max_{1 \le l \le n} |r_n(x_l)| \ge M_n), \\ &\le \sum_{k=1}^K \mathbb{P}(\bar{I}_{n,k}^Z \le \frac{M_n^2}{\epsilon^2}) + \mathbb{P}(\max_{1 \le l \le n} |r_n(x_l)| \ge M_n).\end{aligned}$$

Under Assumption **(A7)**, Lemma 10 Eq. (43) yields $\mathbb{E}[\max_{1 \le l \le n} |r_n(x_l)|] = O(n^{-1/2-\delta})$. Lemma 8 and Markov inequality yield

$$\mathbb{P}(\max_{1 \le k \le K} |\gamma_{n,k}| \ge \epsilon) \le c \left\{ n M_n^{2m} \epsilon^{-2m} + O(n^{-1}) + M_n^{-1} n^{-1/2-\delta} \right\}$$

Setting $M_n = n^{-1/2-\delta}$ yields, for all $m \ge 1$, $\lim_{n \to \infty} \mathbb{P}(\max_{1 \le k \le K} |\gamma_{n,k}| \ge \epsilon) = 0$. (51) finally implies that $\max_{1 \le k \le K} |U_{n,k}^{(1)}| = o_P(1)$. $U_{n,j}^{(2)}$ is a convex combination of $(f_X(x_l) - f_X(y_k))/f_X(y_k)$ ($l \in J_k$) with non-negative weights $I_{n,l}^Z$, thus, (50) yields

$$\max_{1 \le k \le K} |U_{n,k}^{(2)}| \le \max_{1 \le k \le K} \max_{l \in J_k} |f_X(x_l) - f_X(y_k)|/f_X(y_k) = O(n^{-\delta \wedge 1}),$$

which implies that $\mathbb{P}(\max_{1 \le k \le K} U_{n,k}^{(2)} \ge \epsilon) = 0$ for all sufficiently large n. This concludes the proof of Lemma 4.

Appendix C. Proof of Lemma 1

Set $\zeta_{n,k} := \phi'(\bar{I}_{n,k}^Z)$. Under assumption **(S1)** or assumption **(A6)**, (12) and $\mathbb{E}[|Z_0|^\mu] < \infty$ with $\mu \ge \nu_2$, there exists a constant C such that, for all sufficiently large n,

$$\max_{1 \le k \le K} \mathbb{E}\left(\zeta_{n,k}^4\right) \le C. \tag{52}$$

Recall that $\bar{R}_{n,k} = \sum_{j=1}^4 \bar{R}_{n,k}^{(j)}$ and that under **(A7)**, $\mathbb{E}[|\bar{R}_{n,k}^{(j)}|] = o(n^{-1/2})$, $j = 3, 4$. Thus for any reals $\beta_{n,k}$ such that $\sum_{k=1}^K \beta_{n,k}^2 = 1$, applying Hölder inequality, we have

$$\mathbb{E}\left|\sum_{k=1}^K \beta_{n,k} \zeta_{n,k} (\bar{R}_{n,k}^{(3)} + \bar{R}_{n,k}^{(4)})\right| = o(1).$$

To bound the terms involving $\bar{R}^{(j)}_{n,k}$ for $j = 1,2$, we must compute $\mathbb{E}\big[\big(\sum_{k=1}^K \beta_{n,k}\zeta_{n,k}(\bar{R}^{(1)}_{n,k} + \bar{R}^{(2)}_{n,k})\big)^2\big]$. The diagonal terms in the expansion of the square are easily bounded using Lemma 11, (48) and (49)

$$\sum_{k=1}^K \beta_{n,k}^2 \mathbb{E}\left(\zeta^2_{n,k}(\bar{R}^{(1)}_{n,k} + \bar{R}^{(2)}_{n,k})^2\right) \leq \sum_{k=1}^K \beta_{n,k}^2 \left(\mathbb{E}|\zeta_{n,k}|^4\right)^{1/2} \left(\mathbb{E}|\bar{R}^{(1)}_{n,k} + \bar{R}^{(2)}_{n,k}|^4\right)^{1/2} = O(n^{-1}).$$

Consider now the sum over non-diagonal entries

$$V_n = \sum_{1\leq k<k'\leq K} \beta_{n,k}\beta_{n,k'}\mathbb{E}\left(\zeta_{n,k}\zeta_{n,k'}(\bar{R}^{(1)}_{n,k} + \bar{R}^{(2)}_{n,k})(\bar{R}^{(1)}_{n,k'} + \bar{R}^{(2)}_{n,k'})\right).$$

If $\sum_{k=1}^K \beta^2 = 1$, then $\sum_{k\neq k'} \left|\beta_{n,k}\beta_{n,k'}\right| = O(n)$. Thus, to prove that $V_n = o_P(1)$, it is sufficient to prove that for $(i,j) \in \{1,2\}\times\{1,2\}$,

$$\max_{1\leq k\neq k'\leq K} \mathbb{E}\left(\zeta_{n,k}\zeta_{n,k'}\bar{R}^{(i)}_{n,k}\bar{R}^{(j)}_{n,k'}\right) = o(n^{-1}). \tag{53}$$

Using the expression (44) of $\bar{R}^{(j)}_{n,k}$ in terms of $r^{(j)}_{n,l}$, each term $\bar{R}^{(i)}_{n,k}\bar{R}^{(j)}_{n,k'}$ writes as the sum of $4m^2$ terms typified by

$$d^Z_{n,l} r^{(i)}_{n,l} d^Z_{n,l'} r^{(j)}_{n,l'} A(e^{ix_l})A(e^{ix_{l'}})f^{-1}(y_k)f^{-1}(y_{k'}).$$

Since A and f are bounded and bounded away from zero under **(A7)** (and are deterministic functions), (53) will follow from

$$\max_{1\leq k\neq k'\leq K}\ \max_{l\in J_k, l'\in J_{k'}} \left|\mathbb{E}\left(\zeta_{n,k}\zeta_{n,k'}d^Z_{n,l}r^{(i)}_{n,l}d^Z_{n,l'}r^{(j)}_{n,l'}\right)\right| = o(n^{-1}) \tag{54}$$

for $(i,j) = (1,2),(1,1)$ and $(2,2)$.

Note that $\mathbb{E}(r^{(1)}_{n,l}) = 0$ and $r^{(1)}_{n,l}$ is independent from $\zeta_{n,k}, \zeta_{n,k'}, d^Z_{n,l}, d^Z_{n,l'}$ and $r^{(2)}_{n,l'}$. Hence, for all $1 \leq k < k' \leq K$, $l \in J_k$, $l' \in J_{k'}$,

$$\mathbb{E}[\zeta_{n,k}\,\zeta_{n,k'}d^Z_{n,l}r^{(1)}_{n,l}d^Z_{n,l'}r^{(2)}_{n,l'}] = 0, \tag{55}$$

$$\mathbb{E}[\zeta_{n,k}\,\zeta_{n,k'}d^Z_{n,l}r^{(1)}_{n,l}d^Z_{n,l'}r^{(1)}_{n,l'}] = \mathbb{E}[\zeta_{n,k}\,\zeta_{n,k'}d^Z_{n,l}d^Z_{n,l'}]\mathbb{E}[r^{(1)}_{n,l}r^{(1)}_{n,l'}]. \tag{56}$$

(56) and Lemma 10 yield

$$|\mathbb{E}[\zeta_{n,k}\,\zeta_{n,k'}d^Z_{n,l}r^{(1)}_{n,l}d^Z_{n,l'}r^{(1)}_{n,l'}]| \leq cn^{-1}|\mathbb{E}[\zeta_{n,k}\,\zeta_{n,k'}d^Z_{n,l}d^Z_{n,l'}]| \tag{57}$$

This bound is not sufficient to conclude the proof of Lemma 1. We must still prove that $\mathbb{E}[\zeta_{n,k}\zeta_{n,k'}d^Z_{n,l}d^Z_{n,l'}] = o(1)$, uniformly with respect to $k \le K$.

Define, for $\mathbf{w} = (w_1, \ldots, w_{4m}) \in \mathbb{R}^{4m}$,

$$\psi(\mathbf{w}) := \phi'(|\mathbf{w_1}|^2/2)\phi'(|\mathbf{w_2}|^2/2)(w_{2m_1-1} + iw_{2m_1})(w_{2m_2-1} + iw_{2m_2}). \tag{58}$$

where $\mathbf{w_1} = (w_1, \cdots, w_{2m})$, $\mathbf{w_2} = (w_{2m+1}, \cdots, w_{4m})$, $m_1 = l - (k-1)m$, $m_2 = l' - (k'-2)m$. Set $\mathbf{k} = [k(m-1)+1, \cdots, km, k'(m-1)+1, k'm]$. The definitions above imply that

$$\psi(\mathbf{W}_n(\mathbf{k})) = \zeta_{n,k}\zeta_{n,k'}d^Z_{n,l}d^Z_{n,l'}.$$

We now check the validity of an Edgeworth expansion of $\mathbb{E}[\psi(\mathbf{W}_n(\mathbf{k})]$.

- Under **(S1)**, $|\psi(\mathbf{w})| \le C(1 + |\mathbf{w}|^{4\nu+2})$. Since $\mathbb{E}[|Z|^{4\nu+2}] < \infty$, Lemma 6 implies that Edgeworth expansion of $\mathbb{E}[\psi(\mathbf{W}_n(\mathbf{k})]$ is valid up to order 4ν, with a remainder term of order $o(n^{-2\nu}) + O(n^{-(2+4m+1)/2}) = o(n^{-1/2})$.
- Under assumption (13), $\int_{\mathbb{R}^{4m}} |\psi(\mathbf{w})|(1 + |\mathbf{w}|)^{-\nu_3} d\mathbf{w} < \infty$. Since $\mathbb{E}[|Z|^{\nu_3}] < \infty$, Lemma 6 shows that the Edgeworth expansion of $\mathbb{E}[\psi(\mathbf{W}_n(\mathbf{k})]$ is valid up to order ν_3, with a remainder term of order $o(n^{-(\nu_3-2)/2}) = o(n^{-1/2})$.

With the notations of Lemma 6, we can thus write, with $\tilde\nu = 4\nu + 2$ under assumption **(S1)** and $\tilde\nu = \nu_3$ under assumption (13),

$$\mathbb{E}[\psi(\mathbf{W}_n(\mathbf{k})] = \sum_{r=0}^{\tilde\nu} \mathbb{E}_{r,\mathbf{k}}(\psi) + o(n^{-1/2}).$$

It is easily seen that for all $r \le \tilde\nu$, $\max_{1\le k,k'\le K} \mathbb{E}_{r,\mathbf{k}}(\psi) < \infty$, and that

$$\mathbb{E}_{0,\mathbf{k}}(\psi) = \mathbb{E}\left[\phi'(|\xi|^2/2)(\xi_{2m_1-1} + i\xi_{2m_1})\right]\mathbb{E}\left[\phi'(|\xi|^2/2)(\xi_{2m_2-1} + i\xi_{2m_2})\right] = 0.$$

Hence, uniformly in $k, k' \in \{1, \cdots, K\}$ and $l, l' \in J_k \times J_{k'}$,

$$\left|\mathbb{E}\left(\zeta_{n,k}\zeta_{n,k'}d^Z_{n,l}d^Z_{n,l'}\right)\right| = O(n^{-1/2}).$$

This finally implies that $\mathbb{E}\left(\zeta_{n,k}\zeta_{n,k'}d^Z_{n,l'}r^{(1)}_{n,l}d^Z_{n,k'}r^{(1)}_{n,l'}\right) = O(n^{-3/2})$.

There now remains to bound the terms $\mathbb{E}[\zeta_{n,k}\zeta_{n,l}d^Z_{n,l}d^Z_{n,l'}r^{(2)}_{n,l}r^{(2)}_{n,l'}]$. From (40),

$$\mathbb{E}\left(\zeta_{n,k}\zeta_{n,k'}d^Z_{n,l}r^{(2)}_{n,l}d^Z_{n,l'}r^{(2)}_{n,l'}\right) = \frac{1}{2\pi n}\sum_{u,u'} a_u a_v \sum_{v\in I^{(2)}_{n,u}, v'\in I^{(2)}_{n,u'}} \mathbb{E}\left(\zeta_{n,k}\zeta_{n,k'}d^Z_{n,l}d^Z_{n,l'}Z_v Z_{v'}\right)$$

Applying Lemma 9 with $q=2m$, $\mathbf{t}=(t_1,t_2)$, $\mathbf{k}=[(k-1)m+1,\dots,km,(k'-1)m+1,\dots,k'm]$, ψ defined in (58), $r=2$ and $s=\nu_3$, we have,

$$\max_{1\le k,k'\le K}\max_{l\in J_k, l'\in J_{k'}} \left|\mathbb{E}\left(\zeta_{n,k}\zeta_{n,k'}d^Z_{n,l}d^Z_{n,l'}Z_{t_1}Z_{t_2}\right) - \mathbb{E}_0\left(\zeta_{n,k}\zeta_{n,k'}d^Z_{n,l}d^Z_{n,l'}\right)\ \mathbb{E}(Z_{t_1}Z_{t_2})\right| = O(n^{-1/2})$$

Since $\mathbb{E}_0\left(\zeta_{n,k}\zeta_{n,k'}d^Z_{n,l}d^Z_{n,l'}\right)=0$, and $\#[I^{(2)}_{n,u}] = n\wedge|u|$, we get,

$$\sum_{v\in I^{(2)}_{n,u}, v'\in I^{(2)}_{n,u}} \mathbb{E}\left(\zeta_{n,k}\zeta_{n,k'}d^Z_{n,l}d^Z_{n,l'}Z_v Z_{v'}\right) \le cn^{-1/2}(n\wedge|u|)(n\wedge|u'|)$$

Therefore, under **(A7)**,

$$\begin{aligned}\left|\mathbb{E}[\zeta_{n,k}\zeta_{n,k'}d^Z_{n,l}r^{(2)}_{n,l}d^Z_{n,l'}r^{(2)}_{n,l'}]\right| &\le cn^{-1}\sum_{u,v}|a_u||a_v| \sum_{v\in I^{(2)}_{n,u}, v'\in I^{(2)}_{n,u'}} \left|\mathbb{E}[\zeta_{n,k}\zeta_{n,k'}d^Z_{n,l}d^Z_{n,l'}Z_v Z_{v'}]\right| \\ &\le cn^{1/2-2\delta}\sum_{u\in\mathbb{Z}}|u|^\delta|a_u| = o(n^{-1}).\end{aligned}$$

This concludes the proof of Lemma 1.

Appendix D. Proof of Proposition 1

We present an outline of the proof of Proposition 1, which is adapted from Von Sachs (1994) and Janas and Von Sachs (1995). Define $\phi_{m,s}(z) = \phi(z/ms-1)$ and

$$\begin{aligned}\hat\Lambda_{n,x}(s) &= \frac{2\pi}{K}\sum_{k=-K}^{K} W_{b_n}(x-y_k)\phi_{m,s}(\bar I^X_{n,k}),\\ \bar\Lambda_{n,x}(s) &= \frac{2\pi}{K}\sum_{k=-K}^{K} W_{b_n}(x-y_k)\gamma_m(\phi_{m,s}, f_X(y_k)),\\ \bar\Lambda_x(s) &= \gamma_m(\phi_{m,s}, f_X(x))\end{aligned}$$

where

$$\gamma_m(\phi_{m,s},u) = \int \phi(uz/ms-1)g_m(z)dz.$$

Note that, under **(VS2)**, $\hat{\Lambda}_{n,x}(s)$, $\bar{\Lambda}_{n,x}(s)$, $\bar{\Lambda}_x(s)$ are monotone decreasing. In addition $\bar{\Lambda}_x(f_X(x)) = 0$, *i.e.* $f_X(x)$ is a root of the equation $\bar{\Lambda}_x(s) = 0$.

The following lemma is repeatedly used in the sequel. The proof is straightforward and is omitted for brevity.

Lemma 12.

$$\frac{2\pi}{K}\sum_{j=0}^{K} W_{b_n}^p(x-y_j) = b_n^{-(p-1)}\int_{\mathbb{R}} W^p(v)dv(1+o(1)).$$

If v is twice continuously differentiable,

$$\frac{2\pi}{K}\sum_{j=0}^{K} W_{b_n}(x-y_j)v(y_j) = \int_{\mathbb{R}} W_{b_n}(x-y)v(y)dy + O((nb_n^2)^{-1}),$$

$$\int W_{b_n}(x-y)v(y)dy - v(x) = O(b_n^2)$$

Lemma 13. *Assume **(VS1)-(VS3)** and $E|Z|^\mu < \infty$, with $\mu = 8\nu \vee 4$. Then, for all $s \in (-\pi,\pi)\setminus\{0\}$,*

$$\sqrt{Kb_n}(\hat{\Lambda}_{n,x}(s) - \bar{\Lambda}_x(s) \underset{n\to\infty}{\overset{\mathcal{L}}{\Longrightarrow}} \mathcal{N}\big(0, \sigma_W^2\sigma_m^2(\phi_{m,s}, f_X(x))\big), \tag{59}$$

$$\sqrt{Kb_n}(\bar{\Lambda}_{n,x}(s) - \bar{\Lambda}_x(s)) = o(1). \tag{60}$$

Proof of Lemma 13. : Set $\beta_{n,k} := \left(\sum W_{b_n}^2(x-y_k)\right)^{-1/2} W_{b_n}(x-y_k)$. It is easily shown that

$$\frac{2\pi}{K}\sum_k W_{b_n}^2(x-y_k) = b_n^{-1}\int W^2(v)dv(1+o(1)).$$

Hence, $\max_k|\beta_{n,k}| = O((Kb_n)^{-1/2}) = O(\mu_n^{-1/2})$, where $\mu_n = \#\{k : \beta_{n,k} \neq 0\}$ (note that $\mu_n \simeq Kb_n$); it follows that assumptions **(A1)** and **(A2)** are satisfied. These results also imply that

$$n^{-1}\sum|\beta_{n,k}\beta_{n,l}| = O(b_n)$$

Since, $\sup_{\min f_X(y)\le u\le \max f_X(y)} C_m(\phi_{m,s}, u) < \infty$, assumption **(A3)** holds. Finally, under the stated assumptions, $y \to \sigma_m^2(\phi_{m,s}, f_X(y))$ is twice continuously differentiable. Lemma 12 implies that

$$\sum_{k=0}^{K} \beta_{n,k}^2 \sigma_m^2(\phi_{m,s}, f_X(y_k)) = \sigma_m^2(\phi_{m,s}, f_X(x)) + O(b_n^2).$$

and thus assumption **(A4)** is satisfied. We may thus apply Theorem 1 to show that

$$\sum_{k=0}^{K} \beta_{n,k} \phi_{m,s}(\bar{I}_{n,k}^X) - \gamma_m(\phi_{m,s}, f_X(y_k))$$

converges in distribution to a zero-mean Gaussian variable with covariance $\sigma_m^2(\phi_{m,s}, f_X(x))$, showing (59). Eq. (60) is a direct consequence of Lemma 12.

This lemma has the following important corollary

Corollary 1. *Assume **(VS1)**-**(VS3)**. Then,*

$$\sqrt{K b_n}\hat{\Lambda}_{n,x}(f_X(x)) \underset{n\to\infty}{\overset{\mathcal{L}}{\Longrightarrow}} \mathcal{N}\big(0, \sigma_W^2 \sigma_m^2\big). \tag{61}$$

where $\sigma_m^2 = \int \phi^2(z/m - 1) g_m(z) dz$.

Since $\bar{\Lambda}_x(s)$ is monotone decreasing and $\bar{\lambda}_x(f_X(x)) = 0$, $\bar{\Lambda}_x(s)$ is strictly positive/negative for values of s which are strictly smaller/larger than $f_X(x)$. By Lemma 13, for any $0 < \epsilon < \min f_X(x)$, we have

$$\hat{\Lambda}_{n,x}(f_X(x) - \epsilon) \to_P \bar{\Lambda}_x(f_X(x) - \epsilon) > 0,$$
$$\hat{\Lambda}_{n,x}(f_X(x) + \epsilon) \to_P \bar{\Lambda}_x(f_X(x) + \epsilon) < 0,$$

Since $\hat{\Lambda}_{n,x}(s)$ is continuous

$$\lim_{n\to\infty} P(\exists \hat{f}_{n,X}(x) \in [f_X(x) - \epsilon, f_X(x) + \epsilon], \hat{\Lambda}_{n,x}(\hat{f}_{n,X}(x)) = 0) = 1.$$

Since $\hat{\Lambda}_{n,x}(s)$ is monotone decreasing this root (when it exists) is unique. We may thus conclude from this discussion that the equation $\hat{\Lambda}_{n,x}(s) = 0$ has, with a probability tending to 1 as $n \to \infty$ a solution denoted $\hat{f}_{n,X}(x)$, and that these solutions (for different values n) form a consistent sequence of estimators of the spectral density $f_X(x)$. To prove that these estimators (properly normalized) have a limiting distribution, we follow the classical pattern of proof, consisting in linearizing the estimation equation around $f_X(x)$. By the mean value theorem, we have

$$0 = \hat{\Lambda}_{n,x}(\hat{f}_{n,X}(x)) = \hat{\Lambda}_{n,x}(f_X(x)) - \Delta_{n,x}(\hat{f}_{n,X}(x) - f_X(x)) \tag{62}$$

where

$$\Delta_{n,x} = (2\pi/K)\sum_{j=0}^{K} W_{b_n}(x-y_j)\phi'\left(\frac{\bar{I}_{n,j}}{mf_j^*}-1\right)\frac{\bar{I}_{n,j}}{mf_j^{*2}}$$

where $f_j^* = f_X(x) + \eta_j\left(\hat{f}_{n,X}(x) - f_X(x)\right), \eta_j \in [0,1]$. The weak consistency of $\hat{f}_{n,X}(x)$ implies that $\sup_j \left|f_j^* - f_X(x)\right| = o_P(1)$. Using Theorem 1 with $\psi_{m,s}(x) = x\phi'(x/s-1)$ implies

$$\Delta_{n,x} = \frac{\beta_m}{f_X(x)} + o_P(1). \tag{63}$$

Proposition 1 follows from Eq. (62), and (63).

Remark: It is suggested in Von Sachs (1994) that a use of a data taper may be beneficial. The results in Theorem 1 can be adapted to cover that case, under basically the same assumptions (on the data taper) needed to prove Theorem 3.2 (Von Sachs, 1994).

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