Incomplete markets with jumps and Informed agents

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Abstract

An asset is considered whose logarithmic price is the sum of a drift term, a Brownian motion and jumps of a Poisson process. The optimal attainable wealth of both informed and uninformed agents are compared. Various items of future information about the price process are considered available to the informed agent. Detailed analysis is made of the case where the informed agent knows the total number of jumps.

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1 The model

We study a market where are traded

- a riskless asset
- a risky asset.

We assume that the riskless asset has a zero return rate so $S^0_t = 1, \forall t \geq 0$. The dynamics of the risky asset are given by the equation

$$dS_t = S_{t-}(\mu dt + \sigma dW_t + \phi dM_t) \tag{1.1}$$

Here $W = (W_t, t \geq 0)$ is a Brownian motion and $(M_t, t \geq 0)$ is the compensated martingale of a Poisson process, i.e., $M_t = N_t - \lambda t$ where the intensity of the

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Poisson process N is assumed to be a constant $\lambda > 0$. We recall that M and W are independent, as are any pair of BM and PP on the same filtered space. The coefficients μ, σ, ϕ are constants and $\phi > -1$ in order that the price remains non-negative.
The solution of (1.1) is

$$S_t = S_0 \exp(\mu t) \exp(\sigma W_t - \frac{\sigma^2}{2} t + N_t \ln(1+\phi) - \lambda \phi t) . \tag{1.2}$$

This market is incomplete and, in general, the uninformed agent must consider a range of viable prices for any non-hedgeable contingent claim ζ, i.e., consider all the set of expectations $E_Q(\zeta)$ of the terminal payoff under any equivalent martingale measure Q. This set is an interval and is is quite large in general. For example, denoting by $\mathcal{BS}(t,x)$ the Black and Scholes function

$$\mathcal{BS}(t,x) \stackrel{def}{=} E((X_T - K)^+ | X_t = x)$$

where $dX_t = X_t \sigma dW_t$, it is proved in [1] that, for a European option, the range of viable prices is the open interval $]\mathcal{BS}(t,S_t), S_t[$.
Write $(\mathcal{F}_t, 0 \le t \le T)$ for the complete, right continuous filtration generated by S; this is the same as the filtration generated by W and M.

Let us now consider, as in [4] [6] [9] among others, an informed agent who, from time 0, knows, for example, any one of the following pieces of information :
1. The number of jumps over the interval $[0,T]$, i.e., N_T;
2. The times when the jumps occur, i.e., $(N_s, s \le T)$;
3. The path trajectory of the B.M., i.e., $(W_s, s \le T)$;
4. The terminal value of the Brownian motion W, i.e., W_T;
5. The terminal value of the underlying asset, i.e., S_T;
6. Any $\mathcal{F}_T$ random variable ζ.

We do not claim that this model is a "real world" model. We just try to understand better the nature of inside information, modeled as an enlarged filtration.

As was noted in [4] [6] [9], in an incomplete market, when the informed agent knows any *hedgeable* $\mathcal{F}_T$ random variable, there does not exist an equivalent probability measure for the informed agent. From a financial point of view, this is obvious; the informed agent knows the price of the hedgeable ζ contingent claim and he immediately obtains an arbitrage opportunity. This arbitrage opportunity is revealed only at maturity, in the case where ζ is not included in $\mathcal{F}_t$ for $t < T$ (for example $\zeta = S_T$). Therefore, the optimization problem for such an informed agent has no solution.

In [6], the authors study a smaller enlargement of the filtration, as the market is driven by two Brownian noises, and the informed agent knows only the terminal value of one of the BM. More precisely, they assume that the dynamics of the price are given by :

$$dS_t = S_t(\mu_t dt + \sigma_1(t)dW_1(t) + \sigma_2(t)dW_2(t))$$

and the informed agent knows $W_1(T)$, which is not hedgeable. The filtration is the filtration generated by the two BM. They conclude that, if $\sigma_i, i = 1, 2$ are deterministic functions, the non-informed agent and the informed one assign the same price to contingent claims of the form $h(S_T)$, which are hedgeable for any h. This is obvious if one notes that for the uninformed agent

$$dS_t = S_t[\mu_t dt + \sigma_1(t)dW_1(t) + \sigma_2(t)dW_2(t)] = S_t[\mu_t dt + \sigma_3(t)dW_3(t)]$$

where W_3 is a $\mathcal{F}_t$-Brownian motion. Therefore, the market is complete if we consider only contingent claims measurable with respect to $\sigma(S_t, t \leq T)$. Moreover, in the filtration $\mathcal{F}_t^* \stackrel{def}{=} \mathcal{F}_t \vee \sigma(W_1(T))$, the process

$$W_1^*(t) \stackrel{def}{=} W_1(t) - \int_0^t \frac{W_1(T) - W_1(s)}{T - s} ds \stackrel{def}{=} W_1(t) - \int_0^t \Gamma_s ds$$

is a Brownian motion. For the informed agent

$$\begin{aligned} dS_t &= S_t[(\mu_t + \sigma_1(t)\Gamma_t)dt + \sigma_1(t)dW_1^*(t) + \sigma_2(t)dW_2(t)] \\ &= S_t[\ (\mu_t + \sigma_1(t)\Gamma_t)dt + \sigma_3(t)dW_3^*(t)]\,. \end{aligned}$$

Therefore, under the risk neutral probability for the uninformed agent $dS_t = S_t\sigma_3(t)d\tilde{W}_t$ and for the informed agent, under the appropriate risk neutral probability, which is proved to exist, $dS_t = S_t\sigma_3(t)d\tilde{W}_t^*$. The dynamics of the price are the same, and the prices are equal. This does not remain true if the coefficients σ are stochastic, except in the case where they are of the form $\sigma_t = \sigma(t, S_t)$.

1.1 Some particular cases

In this section, we assume that $\sigma \neq 0$ and $\phi \neq 0$.

• In case 2, for the informed agent, the value of the underlying risky asset is lognormal with known jumps at known times. These jumps affect the return of the risky asset. If $\mathcal{F}_t^2 \stackrel{def}{=} \sigma(W_s, s \leq t, N_s, s \leq T)$, any $(\mathcal{F}_t^2)$ martingale is a stochastic integral with respect to the Brownian motion W and is continuous; since S is discontinuous, there does not exist an equivalent martingale measure. This is obvious from a financial point of view: the arbitrage opportunity comes from the knowledge of the times of jumps, and from the fact that the agent knows if the jump is a positive or a negative one. In the case of positive jumps, the agent can buy low before and sell high after the jump. In the particular case $N_T = 0$, we are reduced to the study of case 1, which will be discussed in sections 2, 3 and 4.

• In case 3, let $\mathcal{F}_t^4 \stackrel{def}{=} \sigma(W_s, s \le T, N_s, s \le t)$. If there were an equivalent martingale measure for the informed agent, the dynamics of the price would be of the form $dS_t = S_{t-}\phi d\widetilde{M}^*(t)$, where $\widetilde{M}^*(t)$ is a compensated martingale. Under the historical probability the drift part would be a finite variation process, which is not the case. In the case of positive jumps, the informed agent would be able to know the infimum m of $S_0 \exp(\mu t)\exp(\sigma W_t - \frac{\sigma^2}{2}t - \lambda\phi t)$. Therefore, he knows that $S_t \ge m(1+\phi)^{N_t}$ and he can buy at the last lowest price before T and sell at a high.
Similarly, in case $-1 < \phi < 0$, the informed agent knows the supremum M of $S_0 \exp(\mu t)\exp(\sigma W_t - \frac{\sigma^2}{2}t - \lambda\phi t)$. Suppose this occurs at t^*. Then, for $t^* < t < T$,

$$S_t \le M(1+\phi)^{N_t} \le S_{t^*}(1+\phi)^{N_t - N_{t^*}} \le S_{t^*} \,.$$

Therefore, the informed agent knows he should sell at time t^* and buy at any later time.

• In case 4 we suppose the informed agent knows W_T. Write $K = S_0 e^{\mu T}\exp(\sigma W_T - \frac{\sigma^2}{2}T - \lambda\phi T)$. Then $S_T = K(1+\phi)^{N_T}$. Suppose $\phi > 0$, then $S_T \ge K$. Write $A = \{\omega : \inf_{0\le t\le T} S_t(\omega) < K\}$. Then $P(A) > 0$. If $\omega \in A$ and $S_{t^*}(\omega) < K$, the informed agent would buy at time t^* and sell at T; consequently, there is an arbitrage opportunity.
Suppose $\phi < 0$, then $S_T \le K$. Write $B = \{\omega : \sup_{0\le t\le T} S_t(\omega) > K\}$. Then $P(B) > 0$. If $\omega \in B$ and $S_{t^*} > K$, the informed agent would sell the asset at time t^* and buy at T; again there is an arbitrage opportunity.

• In case 5, there is obviously an arbitrage opportunity on the interval $[0,T]$, but there are no arbitrage opportunities on $[0,t]$ with $t < T$. This case is studied in [4], as well as case 6.

2 Poisson bridge

In the remaining sections of the paper we restrict attention to case 1, that is we suppose the informed agent knows N_T from time 0. Denote by $(\mathcal{F}_t)$ the filtration generated by the price of the risky asset

$$\mathcal{F}_t \stackrel{def}{=} \sigma(S_s, s \le t) = \sigma(W_s, M_s, s \le t)\,.$$

The uninformed agent can use only portfolios which are measurable with respect to $(\mathcal{F}_t)$. The informed agent will use the enlarged filtration

$$\mathcal{F}_t^* \stackrel{def}{=} \mathcal{F}_t \vee \sigma(N_T)\,.$$

As in [8], [7] we can establish that

$$M_t^* \stackrel{def}{=} N_t - \int_0^t \frac{N_T - N_s}{T-s} ds$$

is an $\mathcal{F}_t^*$-martingale. Therefore, for the informed agent, the process N is a point process with the stochastic $(\mathcal{F}_t^*)$-predictable intensity

$$\Lambda_s \stackrel{def}{=} \frac{N_T - N_{s-}}{T-s} .$$

This intensity is zero after the $(\mathcal{F}_t^*)$-stopping time $\tau^* \stackrel{def}{=} \inf\{s \leq T : N_s = N_T\}$. Therefore

$$\int_0^T \frac{N_T - N_s}{T-s} ds = \int_0^{\tau^*} \frac{N_T - N_s}{T-s} ds < \infty, \quad a.s.$$

Note that, for the informed agent, the jumps occur with a hypergeometric distribution. For example, if the agent knows that $N_T = 1$, the single jump occurs with a uniform law on $[0, T]$. Furthermore, if the agent knows that $N_T = n$, as soon as he (or she) observes the occurence of the nth jump, he (or she) knows that there are no remaining jumps and the market is complete for him or her. Consequently, for the informed agent, the dynamics of the price are

$$dS_t = S_{t-}[\mu dt + \phi(\Lambda_t - \lambda)dt + \sigma dW_t + \phi dM_t^*] ,$$

where

$$dM_t^* \stackrel{def}{=} dM_t - (\Lambda_t - \lambda)dt = dN_t - \Lambda_t dt .$$

3 The set of equivalent martingale measures

3.1 The uninformed agent

It was proved, in [1] and [2] for example, that the set of equivalent martingale measures is $\mathcal{Q} = \{P^\gamma | \frac{dP^\gamma}{dP}|_{\mathcal{F}_t} = L_t^\gamma\}$ where $L_t^\gamma = L_{t-}^\gamma(\psi_t dW_t + \gamma_t dM_t)$. In these formulae, the two predictable processes ψ and γ are related by

$$\mu + \sigma\psi_t + \lambda\phi\gamma_t = 0 \ \ , \ \ dP \otimes dt.p.s. \tag{3.1}$$

and the process γ must satisfy $(1 + \gamma_t) > 0$. Note that in the case $\sigma = 0$, the existence of γ requires that $\frac{\mu}{\lambda\phi} < 1$. In fact, if $\phi > 0$ and $\frac{\mu}{\lambda\phi} \geq 1$, the risky asset has a return rate $\mu - \lambda\phi \geq 0 = r$ and the jumps increase the value of this asset. The arbitrage opportunity is obvious. In what follows, we assume that $\mu - \lambda\phi \neq 0$ if $\sigma = 0$.

Under P^γ,

$$W^\gamma(t) \stackrel{def}{=} W(t) - \int_0^t \psi_s \, ds$$

is a Brownian motion and

$$M^\gamma(t) \stackrel{def}{=} M(t) - \int_0^t \lambda\gamma_s ds = N_t - \int_0^t \lambda(1+\gamma_s)ds$$

is a martingale. The dynamics of S, in terms of W^γ and M^γ are given by

$$dS_t = S_{t-}(\sigma dW_t^\gamma + \phi dM_t^\gamma)\,,$$

so

$$S_t = S_0 \mathcal{E}(\sigma W^\gamma)_t \mathcal{E}(\phi M^\gamma)_t\,.$$

Here $\mathcal{E}(\sigma W^\gamma)$ (resp. $\mathcal{E}(\phi M^\gamma)$) is the Doléans-Dade exponential, i.e., the solution of $dX_t = X_t \sigma dW_t^\gamma, X_0 = 1$ (resp. $dX_t = X_{t-}\phi dM_t^\gamma$, $X_0 = 1$).

The range of European option prices is the open interval $]\mathcal{BS}(t,S_t), S_t[$.

3.2 The informed agent

3.2.1 Equivalent martingale measures for the informed agent

Recall that for the informed agent,

$$dS_t = S_{t-}[\mu dt + \phi(\Lambda_t - \lambda)dt + \sigma dW_t + \phi dM_t^*]$$

The pair (W, M^*) has the predictable representation property. Therefore, if $P^{*\gamma}$ is any equivalent martingale measure for the informed agent, the Radon-Nykodym density of $P^{*\gamma}$ with respect to P is a P martingale $L^{*\gamma}$ which admits a representation of the form

$$dL_t^{*\gamma} = L_{t-}^{*\gamma}(\psi_t^* dW_t + \gamma_t^* dM_t^*)$$

The process S is a $P^{*\gamma}$-$(\mathcal{F}_t^*)$-martingale if and only if $(SL^{*\gamma})$ is a P-$(\mathcal{F}_t^*)$-martingale, which is equivalent to

$$\mu + \sigma\psi_t^* - \lambda\phi + \phi\Lambda_t(1+\gamma_t^*) = 0 \quad , \quad dP \otimes dt.p.s. \tag{3.2}$$

Under $P^{*\gamma}$, the process $(W_t^{*\gamma} \stackrel{def}{=} W_t - \int_0^t \psi_s^* ds, t \geq 0)$ is a Brownian motion and $(M_t^{*\gamma} = N_t - \int_0^t \Lambda_s(\gamma_s^* + 1)\, ds, t \geq 0)$ is a martingale. Unfortunately, the independence of these processes is lost under P^* because ψ^* depends on N.

Note that when $\sigma = 0$ there does not exist a process γ such that (3.2) is satisfied, as for $t > \tau^*$, $\Lambda_t = 0$ and (3.2) reduces to $\mu - \lambda\phi = 0$. We have studied this case in the previous section, proving that there is an arbitrage. In the case $\mu - \lambda\phi \neq 0$ we can easily check that there is an arbitrage opportunity: as soon as the informed agent knows that there will be no more jumps, the risky asset is riskless with return rate $\mu - \lambda\phi$. If this return differs from the return of the riskless asset, clearly there is an arbitrage opportunity.

For the informed agent, the price of the underlying asset is, in the general case

$$dS_t = S_{t-}(\sigma dW_t^{*\gamma} + \phi dM_t^{*\gamma}) \,,$$

and

$$S_t = S_0 \mathcal{E}(\sigma W^{*\gamma})_t \, \mathcal{E}(\phi M^{*\gamma})_t \,.$$

3.2.2 Range of prices for the informed agent

Definition : In the case $r = 0$, we define the Black-Scholes function $\mathcal{BS}(t,x)$ by

$$\mathcal{BS}(t,x) = E(\,(X_T - K)^+ \,|X_t = x)\ ,\ \ \mathcal{BS}(T,x) = \ (x-K)^+$$

when the dynamics of X are given by

$$dX_t = X_t \sigma \, dW_t \,, \tag{3.3}$$

i.e. $\mathcal{BS}(t,x) = E[(x \exp[\sigma\sqrt{T-t}\,U - \frac{1}{2}\sigma^2(T-t)] \ - K)^+]$ where U is a standard normal random variable.

As in [1], it is easy to establish that the price of a European claim is bounded below by the Black-Scholes function evaluated at the value of the asset, i.e. $\mathcal{BS}(t, S_t)$: Itô's formula and the PDE satisfied by $\mathcal{BS}$ yield

$$\mathcal{BS}(t,S_t) = \mathcal{BS}(0,S_0) + \int_0^t \Big[\Upsilon \mathcal{BS}(s,S_s)\Lambda_s(1+\gamma_s^*)\Big] ds + Z_t \,.$$

Here $\Upsilon\mathcal{BS}(t,x) = \mathcal{BS}(t, x(1+\phi)) - \mathcal{BS}(t,x) - \phi x \dfrac{\partial \mathcal{BS}}{\partial x}(t,x)$ and Z is an $\mathcal{F}_t^*$ martingale. From the convexity of the Black-Scholes function, $E^{*\gamma}((S_T-K)^+) \geq \mathcal{BS}(0,S_0)$. The Black and Scholes price is the lower bound of the range of prices, as can be seen when γ goes to -1. As soon as $t > \tau^*$, the range interval is reduced to the Black-Scholes price, because we have noticed that, after the last jump, the market is complete for the informed agent.
Before the last jump, the upper bound is equal to the value of the underlying asset, as shown in [5].

4 Optimisation

4.1 A toy market

We restrict our attention to the simple case of a complete market where

$$dS_t = S_{t-}[\mu dt + \phi dM_t]$$

and $r = 0$. We study only the case of the optimisation problem for the uninformed agent, since the informed agent has an arbitrage opportunity after time τ^*. The non-arbitrage condition is equivalent to the existence of γ such that $\gamma > -1$ and $\mu + \lambda\phi\gamma = 0$. This implies that $\dfrac{\lambda\phi - \mu}{\lambda\phi} > 0$. The unique equivalent martingale measure Q is defined by $\dfrac{dQ}{dP}|_{\mathcal{F}_t} = L_t$ where L is the strictly positive martingale

$$L_t = \left(\frac{\lambda\phi - \mu}{\lambda\phi}\right)^{N_t} \exp(t\mu/\phi)$$

which satisfies $dL_t = -L_{t-}\dfrac{\mu}{\lambda\phi}dM_t$. Under the measure Q, $\tilde{M}_t \stackrel{def}{=} M_t + t\mu/\phi$ is a martingale. Suppose α_t and β_t are predictable processes representing the portfolio of the uninformed agent. That is, α_t is the amount of riskless asset owned at time t and β_t is the number of units of S held at time t. The wealth of the investor is then $X_t = \alpha_t + \beta_t S_t$. If $c_t \geq 0$ is an adapted process representing the consumption, X satisfies the self-financing condition $dX_t = \beta_t dS_t - c_t dt = \pi_t X_{t-}(\mu dt + \phi dM_t) - c_t dt$, where $\pi_t = \beta_t S_t / X_t$ is the proportion of the wealth invested in S at time t. The process

$$\left(\int_0^t c_s L_s ds + X_t L_t, t \leq T\right)$$

is a local martingale under the historical measure. Suppose the investor wishes to maximize the expectation

$$E\left(\int_0^T u(c_s)ds + g(X_T)\right)$$

Here u and g are appropriate utility functions, $\tilde{u}$ and $\tilde{g}$ will denote their conjugate functions. Write $X_0 = x$ for the initial wealth. Then the Lagrangian is $E(\int_0^T u(c_s)ds + g(X_T) - \nu(L_T X_T + \int_0^T L_s c_s ds - x))$. The optimal pair is given by

$$\widetilde{c}_t = -\tilde{u}'(\tilde{\nu} L_t), X_T^* = -\tilde{g}'(\tilde{\nu} L_T)$$

with $\tilde{\nu}$ such that the following budget constraint holds : $E\left(L_T \widetilde{X}_T + \int_0^T L_s \widetilde{c}_s ds\right) = x$. Using the fact that for any β the process $(L_t)^\beta \exp(-\Gamma(\beta)t)$ is a martingale, where

$$\Gamma(\beta) = \lambda[(1 - \frac{\mu}{\lambda\phi})^\beta - 1 + \frac{\beta\mu}{\lambda\phi}]\,,$$

we are able to make explicit the computations in some cases:

• **Log utilities.**

- In the particular case $u(x) = g(x) = \ln x$, we obtain

$$\widetilde{\nu} = \frac{1+T}{x}\,,\ \widetilde{X}_T = \frac{x}{(1+T)L_T},\ \widetilde{c}_t = \frac{x}{(1+T)L_t}\,.$$

The current optimal wealth is $\widetilde{X}_t = \dfrac{x}{L_t}(1 - \dfrac{t}{1+T})$ and the optimal portfolio is $\widetilde{\pi}_t = \dfrac{\mu}{\phi(\lambda\phi - \mu)}$.

- In the case $u(x) = 0, g(x) = \ln(x)$, we obtain $\widetilde{X}_t = x(L_t)^{-1}$ and the same optimal portfolio.

• **Power utility functions.**
Consider the case $u(x) = g(x) = x^\alpha$, with $0 < \alpha < 1$. The optimal consumption is $\widetilde{c}_t = \left(\dfrac{\nu L_t}{\alpha}\right)^\beta$ where $\beta = \dfrac{1}{\alpha - 1}$ and

$$\frac{1}{\nu^\beta} = \frac{1}{x\alpha^\beta}[e^{\Gamma(\beta+1)T} + \frac{1}{\Gamma(\beta+1)}(e^{\Gamma(\beta+1)T} - 1)]$$

The current optimal wealth is

$$\widetilde{X}_t = \frac{x\Phi(T-t)}{\Phi(T)}\,L_t^\beta$$

where $\Phi(\tau) = \Gamma(\beta+1)\,\exp[\Gamma(\beta+1)\tau] + \exp[\Gamma(\beta+1)\tau] - 1$. The optimal portfolio is

$$\widetilde{\pi} = \frac{1}{\phi}[(1 - \frac{\mu}{\lambda\phi})^\beta - 1]$$

These quantities $\widetilde{X}_T$ and consumption strategy $\widetilde{c}$ give the maximum expected utility for the uninformed agent. We have seen that, if the informed investor knows N_T, he (or she) has an arbitrage opportunity and so his (or her) expected potential wealth can be infinite.

4.2 General case

4.2.1 The un-informed agent

We suppose now that the risky asset has dynamics

$$dS_t = S_{t-}\,(\mu dt + \sigma dW_t + \phi dM_t)$$

Let $(X_t, t \geq 0)$ be the wealth of an agent whose portfolio is again described by (π_t), the proportion of wealth invested in the asset S at time t. Then

$$dX_t = \pi_t X_{t-}(\mu dt + \sigma dW_t + \phi dM_t) - c_t dt \qquad (4.1)$$

The market is now incomplete. There are various ways to solve the problem.
One is to complete the market with a second asset, solve the optimization problem in the complete market and adjust the parameters of the fictitious asset such that the optimal portfolio has a zero component in this asset.
A second is to solve directly the problem in the incomplete market by means of the dynamic programming method. Let

$$V(t,x) = \sup_{c,x} E\left[\int_t^T u(c_s)ds + g(X_T^{x,\pi,c})|X_t = x\right]$$

be the value function for this optimal control problem, representing the maximum attainable expected utility. The process $V(t,X_t) + \int_0^t u(c_s)ds$ is a supermartingale and a martingale for optimal wealth. Therefore, the value function V is a solution of the HJB equation

$$\begin{aligned} V(T,x) &= g(x) \\ 0 &= \frac{\partial V}{\partial t}(t,x) + \sup_{\pi,c}\{u(c) + \frac{\partial V}{\partial x}(t,x)(\pi x\mu - c) + \frac{1}{2}\sigma^2\pi^2x^2\frac{\partial^2 V}{\partial x^2}(t,x) \\ &\quad + \lambda[V(t,x(1+\pi\phi)) - V(t,x) - \pi x\phi\frac{\partial V}{\partial x}(t,x)]\} \end{aligned}$$

This problem can be solved explicitly for some particular cases.

4.2.2 Logarithmic utilities

• Case: $g(x) = \ln(x), u(x) = 0$
A solution of the HJB equation is of the form $V(t,x) = p(t)\ln x + q(t)$ with $p'(t) = 0$, $p(T) = 1$, $q'(t) + pm = 0$, $q(T) = 0$ where $m = \sup_\pi \pi\mu + \lambda[\ln(1+\pi\phi) - \pi\phi] - \frac{1}{2}\pi^2\sigma^2$, the supremum being reached for $\tilde{\pi}$ satisfying $\mu - \lambda\phi^2[\frac{\tilde{\pi}}{1+\tilde{\pi}\phi}] - \tilde{\pi}\sigma^2 = 0$ and $1 + \tilde{\pi}\phi > -1$.
Then $V(t,x) = \ln x + m(T-t)$. In particular,

$$V(0,x) = E(\ln(\widetilde{X}_T)) = \ln x + \frac{1}{2}\sigma^2\tilde{\pi}^2T + \lambda\ln(1+\phi\tilde{\pi})T + \lambda(\frac{1}{1+\phi\tilde{\pi}} - 1)T\,.$$

This gives the maximum expected utility which can be attained by the uninformed agent.
For this case, there is in fact a third way to solve the problem: determine $E(\ln(X_T))$ using Itô's formula. This leads to

$$E[\ln(X_T)] = \ln(x) + \int_0^T E(\mu\pi_s - \frac{1}{2}\sigma^2\pi_s^2 + \lambda(\ln(1+\phi\pi_s) - \phi\pi_s)ds|,.$$

We theen maximize the quantity under the integral sign for each s and ω. Any of the methods leads to

$$\tilde{\pi} = \frac{1}{2\sigma^2\phi}\left(\mu\phi - \phi^2\lambda - \sigma^2 \pm \sqrt{(\mu\phi - \phi^2\lambda - \sigma^2)^2 + 4\sigma^2\phi\mu}\right) ,$$

where the quantity under the square root is equal to $(\mu\phi - \phi^2\lambda + \sigma^2)^2 + 4\sigma^2\phi^2\lambda$ and, therefore, is non-negative. The sign to be used depends on the sign of quantities related to the parameters. The optimal $\tilde{\pi}$ is the only one such that $1 + \phi\pi > 0$. Solving the equation (4.1), it can be proved that the optimal wealth is $\widetilde{X}_t = x(\widetilde{L}_t)^{-1}$ where $d\widetilde{L}_t = \widetilde{L}_{t-}(-\sigma\tilde{\pi}dW_t + (\frac{1}{1+\phi\tilde{\pi}} - 1)dM_t)$ is a Radon Nykodym density of an equivalent martingale measure. In this incomplete market, we thus obtain the utility equivalent martingale measure defined by Davis [3].

• Case $u(x) = \ln x\,, g(x) = \ln x$. A solution of the HJB equation is of the form $V(t,x) = p(t)\ln x + q(t)$ where $p(t) = 1 + T - t$ and $q'(t) - \ln p(t) - 1 + mp(t) = 0, q(T) = 0$ so that

$$q(t) = \frac{m}{2}(T-t)^2 + m(T-t) - (1+T-t)\ln(1+T-t)$$

The optimal portfolio is the same as in the previous case, the optimal wealth is $\widetilde{X}_t = \frac{x}{\widetilde{L}_t}(1 - \frac{t}{1+T})$, with the same $\widetilde{L}$ and the optimal consumption is $\tilde{c}_t = \frac{\widetilde{X}_t}{1+T-t}$.

4.2.3 Power utilities

• Case $g(x) = x^\alpha, u(x) = 0, 0 < \alpha < 1$. A solution is of the form $V(t,x) = p(t)x^\alpha$ with $p(T) = 1, p'(t) + Mp(t) = 0$ where $M = \sup_\pi\{\alpha\pi\mu + \lambda[(1+\pi\phi)^\alpha - 1 - \pi\phi\alpha] + \frac{1}{2}\pi^2\sigma^2\alpha(\alpha-1)\}$, the supremum being reached for $\tilde{\pi}$ satisfying $\mu + \lambda\phi[(1+\tilde{\pi}\phi)^{\alpha-1} - 1] + (\alpha-1)\tilde{\pi}\sigma^2 = 0$ and $1 + \tilde{\pi}\phi > -1$. Then, $V(t,x) = e^{-M(t-T)}x^\alpha$. The optimal wealth is $\widetilde{X}_t = x\Phi(t)\left[\widetilde{L}_t^\gamma\right]^\beta$, with $\beta = 1/(\alpha-1)$, $\gamma = (1+\phi\tilde{\pi})^{\alpha-1} - 1$ and

$$\Phi(t) = \exp At,\ A = \tilde{\pi}^2\sigma^2(1 - \frac{1}{2}\alpha) + \lambda(1+\tilde{\pi}\phi)^{\alpha-1}((\alpha-1)\phi + 1) + \frac{\lambda}{\alpha-1}\,.$$

• Case $u(x) = x^\alpha = g(x)$. A solution is of the form $V(t,x) = p(t)x^\alpha$, so $\tilde{c} = x[p(t)]^\beta$ where $\beta = 1/(\alpha-1)$ and $p(t) = \left(\frac{M}{(\alpha-1)(1-k\exp(\beta Mt))}\right)^{\alpha-1}$ where $k = (1 - \frac{M}{\alpha-1})\exp(-\beta MT)$.

4.2.4 The informed agent

As we are considering case 1, the informed agent knows N_T from time 0. Therefore, his wealth evolves according to the dynamics

$$dX_t^* = \pi_t X_{t-}^*[(\mu + \phi(\Lambda_t - \lambda)]dt + \sigma dW_t + \phi dM_t^*]$$

Exactly the same computations as above can be carried out. In fact these result in changing μ to $(\mu + \phi(\Lambda_t - \lambda)$ and the intensity of the jumps from λ to Λ_t.

• Case $g(x) = \ln(x), u(x) = 0$
The optimal portfolio π^* is now such that $\mu - \lambda\phi + \phi\Lambda_s[\frac{1}{1+\pi^*\phi}] - \pi^*\sigma^2 = 0$ and is given by

$$\pi_s^* = \frac{1}{2\sigma^2\phi}\left(\mu\phi - \phi^2\lambda - \sigma^2 \pm \sqrt{(\mu\phi - \phi^2\lambda + \sigma^2)^2 + 4\sigma^2\phi^2\Lambda_s}\right) ,$$

The optimal wealth is $X_t^* = x(L_t^*)^{-1}$ where

$$dL_t^* = L_{t-}^*(-\sigma\pi_s^* dW_t + (\frac{1}{1+\phi\pi_s^*} - 1)dM_t^*)$$

Whereas the optimal portfolio of the uninformed agent is a constant one the optimal portfolio of the informed agent is time-varying and has a jump as soon as a jump occurs for the prices.
The maximum attainable wealth for the uninformed agent is obtained using the constant strategy $\tilde{\pi}$ for which

$$\tilde{\pi}\mu + \lambda[\ln(1+\tilde{\pi}\phi) - \tilde{\pi}\phi] - \frac{1}{2}\tilde{\pi}^2\sigma^2 = \sup_\pi[\pi\mu + \lambda[\ln(1+\pi\phi) - \pi\phi] - \frac{1}{2}\pi^2\sigma^2]$$

In contrast, the informed agent must maximize at each (s,ω) the quantity

$$\pi\mu + \Lambda_s(\omega)\ln(1+\pi\phi) - \lambda\pi\phi - \frac{1}{2}\pi^2\sigma^2$$

Consequently,

$$\sup_\pi \pi\mu + \Lambda_s\ln(1+\pi\phi) - \lambda\pi\phi - \frac{1}{2}\pi^2\sigma^2 \geq \tilde{\pi}\mu + \Lambda_s\ln(1+\tilde{\pi}\phi) - \lambda\tilde{\pi}\phi - \frac{1}{2}\tilde{\pi}^2\sigma^2$$

Now, $E[\Lambda_s] = \lambda$, so

$$\begin{aligned}\sup_\pi E(\ln X_T^*) &= \ln x + \sup_\pi \int_0^T E(\pi\mu + \Lambda_s\ln(1+\pi\phi) - \lambda\pi\phi - \frac{1}{2}\pi^2\sigma^2)ds \\ &\geq \ln x + \int_0^T \tilde{\pi}(\mu + \lambda\ln(1+\tilde{\pi}\phi) - \lambda\tilde{\pi}\phi - \frac{1}{2}\tilde{\pi}^2\sigma^2)ds = E(\ln \widetilde{X}_T)\end{aligned}$$

Therefore, the maximum expected wealth for the informed agent is greater than that of the uninformed agent. This is obvious because the informed agent can use any strategy available to the uninformed agent.

• case $u(x) = 0\,, g(x) = x^\alpha$
The optimal portfolio π^* of the informed agent is now given by $\mu + \phi[\Lambda(1 + \pi^*\phi)^{\alpha-1} - \lambda] + (\alpha-1)\pi^*\sigma^2 = 0$, and the optimal wealth is $X_t^* = x\Phi^*(t)[L_t^{*\gamma}]^\beta$, with $\beta = 1/(\alpha-1)$, $\gamma_s = (1+\phi\pi_s^*)^{\alpha-1} - 1$ and

$$\Phi(t) = \exp\int_0^t A_s^* ds,\ A_s^* = [\pi_s^*]^2\,\sigma^2(1-\frac{1}{2}\alpha) + \Lambda_s(1+\pi_s^*\phi)^{\alpha-1}[(\alpha-1)\phi+1] + \frac{\Lambda_s}{\alpha-1}$$

5 Signalling

In this section we discuss what the uninformed agent might detect from the actions and investment strategy of the informed agent.
In general banks do not disclose the amounts customers hold in savings. Consequently, although the price process S and the number of shares β of S are observed variables, the total wealth and the proportion π of wealth invested in S are not. Of course $\beta = \frac{\pi X}{S}$ or $\pi = \beta\frac{S}{X}$.

In the models of section 4, with utility functions $\ln x$ or x^α, the optimal $\tilde{\pi}$ for the uninformed agent is constant. However, the optimal β is not constant. If t is a jump time of the risky asset, $X_t = X_{t-}(1+\tilde{\pi}\phi)$ and $S_t = S_{t-}(1+\phi)$, so $\frac{\beta_t S_t}{\beta_{t-}} = (1+\phi\tilde{\pi})$. Consequently, the value of the $\tilde{\pi}$ being used is revealed at jump times. Similarly, for the informed agent, the optimal π^* is not constant but it is continuous between jump times. At a jump time t, the uninformed agent will observe the quantities $\beta_{t-}^* S_{t-}$ and $\beta_t^* S_t$ of the informed agent. If the informed agent is using this optimal π^*

$$\frac{\beta_t^* S_t}{\beta_{t-}^*} = (1+\pi_{t-}^*\phi)\frac{\pi_t^*}{\pi_{t-}^*}$$

which can be written $(1+\theta_{t-}\phi)$ for some θ_{t-}. Consequently, after one jump the uninformed agent might conclude the informed agent is using the strategy θ. However, after observing the action of the informed agent at two jump times two different values of θ will be detected. This may signal to the uninformed agent that the informed agent does have some extra knowledge. If the informed agent knows N_T the number of jumps, a strategy which would avoid disclosing inside knowledge might be for him to use $\tilde{\pi}$ until the time of the last jump and then to use an optimal π^* over the remaining period to T.

6 Conclusion

We have studied various situations in which an informed agent knows future information when a risky asset has dynamics involving both a Brownian motion and a Poisson process. The most interesting cases are those in which the informed agent knows only the final number of jumps of the Poisson process. The maximal attainable wealth for both informed and uninformed agents has then been explicitly calculated.

References

[1] N. Bellamy and M. Jeanblanc. Incompletness of markets driven by a mixed diffusion. *Preprint, Université d'Evry*, 1997.

[2] H. Buhlmann, F. Delbaen, P. Embrecht, and A. Shiryaev. No-arbitrage, change of measure and conditional Essher transform in a semi-martingale model of stock price. *CWI Quarterly*, 9:291–317, 1996.

[3] M.H.A. Davis. Option pricing in Incomplete markets . In M.H.A. Demtser and S.R. Pliska, editors, *Mathematics of Derivative Securities*, Publication of the Newton Institute, pages 216–227. Cambridge University Press, 1997.

[4] L. Denis, A. Grorud, and M Pontier. Formes de Dirichlet, grossissement de filtration et anticipation dans un marché discontinu. *Preprint*, 1997.

[5] E. Eberlein and J. Jacod. On the range of option pricing. *Finance and Stochastic*, 2, 1997.

[6] R. Elliott, H. Geman, and B.M. Korkie. Portfolio optimization and contingent claim pricing with differential information. *Stochastics and stochastics report*, 60:185–203, 1997.

[7] R. Elliott and P.E. Kopp. Equivalent martingale measures for bridge processes. *Stochastic analysis and applications*, 9:429–444, 1991.

[8] T. Jeulin and M. Yor. Inégalité de Hardy, semimartingales et faux-amis . In P.A. Meyer, editor, *Séminaire de probabilités*, Lecture notes in Mathematics, 721, pages 332–359. Springer-Verlag, 1979.

[9] I. Pikovsky and I. Karatzas. Anticipative portfolio optimization. *Finance and stochastics, Forthcoming*, 1998.