Multiple stochastic integral expansions of arbitrary Poisson jump times functionals

Nicolas Privault

Equipe d'Analyse et Probabilités, Université d'Evry-Val d'Essonne
Boulevard des Coquibus, 91025 Evry Cedex, France
e-mail: privault@lami.univ-evry.fr

Abstract

We compute the Wiener-Poisson expansion of square-integrable functionals of a finite number of Poisson jump times in series of multiple Poisson stochastic integrals.

Key words: Fock space, Poisson process, Wiener chaos.

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1 Introduction

Any square-integrable functional on the Wiener, resp. Poisson space can be expanded into a series of multiple stochastic integrals with respect to the Wiener, resp. Poisson process. This property is known as the Wiener chaos representation property. On the Wiener space, the chaos expansion of a functional of d independent single stochastic integrals can be computed using Wiener-Hermite orthogonal expansions. More generally, the gradient operator on Fock space allows to compute the expansion of certain square-integrable functionals, cf. [8]. However, on the Poisson space this gradient is identified to a finite difference operator whose repeated application leads to complicated expressions. In [7] an induction relation was obtained and used to compute the expansion of the Poisson process jump times on $\mathbb{R}_+$, using the Clark-Ocone and Stroock formulas associated to different Poisson gradient operators. A direct calculation using only the formula of [8] and the Fock gradient can be found in [3], concerning a Poisson process on a bounded interval.

In this paper, these results are extended to general square-integrable functions of a finite number of Poisson jump times. This allows in particular to compute the chaos expansions of solutions of adapted stochastic differential equations driven by a standard Poisson process, since the value at time t of such solutions can be expressed as functionals of finite numbers of jump times. Our proof is elementary in the sense that is uses only Poisson-Charlier orthogonal expansions instead of the gradient on Fock space. The results obtained on Poisson space are compared to their counterpart on the Gaussian space and a class of exponential vectors on Fock space, which are stable under multiplication on Poisson space, is constructed.

2 Notation

In this section we recall some facts concerning Charlier polynomials, Poisson multiple stochastic integrals and the Fock space, cf. [9]. Let $(N_t)_{t\in\mathbb{R}_+}$ denote a (right-continuous) standard Poisson process on the real line, with jump times $(T_n)_{n\geq 1}$. Let $(T_k)_{k\geq 1}$ denote the increasing family of jump times of $(N_t)_{t\in\mathbb{R}_+}$. The underlying probability space is denoted by Ω, so that $L^2(\Omega)$ is the space of square-integrable functionals of $(N_t)_{t\in\mathbb{R}_+}$. For $n\in\mathbb{Z}$ and $t\in\mathbb{R}_+$, let

$$p_n^t = \begin{cases} \frac{t^n}{n!}e^{-t}, & \text{if } n\geq 0,\\ 0, & \text{if } n<0.\end{cases}$$

For fixed $n\geq 1$, $p_{n-1}:\mathbb{R}\to\mathbb{R}_+$ is the density function of the n-th jump time T_n of $(N_t)_{t\in\mathbb{R}_+}$. On the other hand, for fixed $t\in\mathbb{R}_+$, $p^t:\mathbb{Z}\to\mathbb{R}_+$ is the discrete distribution of the random variable N_t. The Charlier polynomial of order $n\in\mathbb{N}$ and parameter $t\in\mathbb{R}_+$ is defined as

$$C_n^t(x) = \frac{(-1)^n}{p_x^t}t^n(\Delta^x)^n p_x^t, \qquad x\in\mathbb{Z},$$

where Δ^x is the finite difference operator $\Delta^x f(x) = f(x) - f(x-1)$. We have the relation $\partial p_x^t = -\Delta^x p_x^t$, where ∂ denotes differentiation with respect to t, hence we may also write

$$C_n^t(x) = \frac{1}{p_x^t}t^n\partial^n p_x^t, \qquad x\in\mathbb{R}_+. \tag{1}$$

For every $t\in\mathbb{R}_+$, the family $(C_n^t)_{n\in\mathbb{N}}$ is orthogonal in $l^2(\mathbb{Z},p^t)$ and the square norm of C_n^t is $n!t^n$. We denote by $L^2(\mathbb{R}_+^{\circ n})$ the space of square-integrable and symmetric functions on $\mathbb{R}^n$, with norm $\|\cdot\|^2_{L^2(\mathbb{R}_+)^{\circ n}} = n!\ \|\cdot\|^2_{L^2(\mathbb{R}_+^n)}$, and by $f_n\circ g_m$ the symmetric tensor product of $f_n\in L^2(\mathbb{R}_+)^{\circ n}$ and $g_m\in L^2(\mathbb{R}_+)^{\circ m}$. Let $0\leq t_1\leq\cdots\leq t_d$ and $k_1,\ldots,k_d\in\mathbb{N}$. The multiple Poisson stochastic integral of the function

$1_{[0,t_1]}^{\circ k_1} \circ 1_{[t_1,t_2]}^{\circ k_2} \circ \cdots \circ 1_{[t_{d-1},t_d]}^{\circ k_d}$ is defined as

$$I_n\left(1_{[0,t_1]}^{\circ k_1} \circ 1_{[t_1,t_2]}^{\circ k_2} \circ \cdots \circ 1_{[t_{d-1},,t_d]}^{\circ k_d}\right) = C_{k_1}^{t_1}(N_{t_1}) C_{k_2}^{t_2-t_1}(N_{t_2}-N_{t_1}) \cdots C_{k_d}^{t_d-t_{d-1}}(N_{t_d}-N_{t_{d-1}}), \tag{2}$$

and this expression is extended to symmetric square-integrable functions as

$$I_n(f_n) = n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1,\ldots,t_n) d\tilde{N}_{t_1} \cdots d\tilde{N}_{t_n},$$

with $\tilde{N}_t = N_t - t$, $t \in \mathbb{R}_+$, cf. [2], [9]. We have the isometry

$$(I_n(f_n), I_m(g_m))_{L^2(\Omega)} = 1_{\{n=m\}}(f_n, g_m)_{L^2(\mathbb{R}_+)^{\circ n}}, \quad f_n \in L^2(\mathbb{R}_+)^{\circ n},\ g_m \in L^2(\mathbb{R}_+)^{\circ m},$$

which extends the norm properties of Charlier polynomials. If $f_n \in L^2(\mathbb{R}_+^n)$ is not symmetric we let $I_n(f_n) = I_n(\tilde{f}_n)$, where $\tilde{f}_n$ is the symmetrization of f_n, defined as

$$\tilde{f}_n(t_1,\ldots,t_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} f(t_{\sigma(1)},\ldots,t_{\sigma_n}),$$

where Σ_n is the set of all permutations of $\{1,\ldots,n\}$. For precision of notation we will often write explicitly the variables $t_1,\ldots,t_n$ in $I_n(f_n(t_1,\ldots,t_n))$.
The Fock space

$$\Gamma(L^2(\mathbb{R}_+)) = \bigoplus_{n \geq 0} L^2(\mathbb{R}_+)^{\circ n}$$

is identified to $L^2(\Omega)$ via multiple stochastic integrals of symmetric square-integrable functions. On $\Gamma(L^2(\mathbb{R}_+))$ are defined the annihilation and creation operators $D : \Gamma(L^2(\mathbb{R}_+)) \to \Gamma(L^2(\mathbb{R}_+)) \otimes L^2(\mathbb{R}_+)$ and $\delta : \Gamma(L^2(\mathbb{R}_+)) \otimes L^2(\mathbb{R}_+) \to \Gamma(L^2(\mathbb{R}_+))$ by $\delta(f^{\circ n} \otimes g) = g \circ f^{\circ n}$, $f, g \in L^2(\mathbb{R}_+)$ and $Df^{\circ n} = nf \otimes f^{\circ(n-1)}$. We note that D is identified to a finite difference operator, cf. [5], [6]. The Wick exponential $\varepsilon(u)$ is defined as

$$\varepsilon(u) = \sum_{n \geq 0} \frac{1}{n!} u^{\circ n}.$$

Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing $\mathcal{C}^\infty$ functions.

3 Chaos expansions of jump times functionals

Let

$$\Delta_n = \{(t_1,\ldots,t_n) \in \mathbb{R}_+^n \ :\ t_1 \leq \cdots \leq t_n\}, \quad n \geq 1.$$

For $f \in \mathcal{C}^1(\mathbb{R}^d)$, $\partial_i f$ represents the partial derivative of f with respect to its i-th variable, $1 \leq i \leq d$. We first state a result for smooth functions of a finite number of jump times. As a convention, if $k_1 \geq 0, \ldots, k_d \geq 0$ satisfy $k_1 + \cdots + k_d = n$, we define $(t^1_1, \ldots, t^1_{k_1}, t^2_1, \ldots, t^2_{k_2}, \ldots, t^d_1, \ldots, t^d_{k_d})$ as

$$(t^1_1, \ldots, t^1_{k_1}, t^2_1, \ldots, t^2_{k_2}, \ldots, t^d_1, \ldots, t^d_{k_d}) = (t_1, \ldots, t_n).$$

Theorem 1 *Let $n_1, \ldots, n_d \in \mathbb{N}$ with $1 \leq n_1 < \cdots < n_d$, and let $f \in \mathcal{C}^d_c(\Delta_d)$. The chaos expansion of $f(T_{n_1}, \ldots, T_{n_d})$ is given as*

$$f(T_{n_1}, \ldots, T_{n_d}) \quad = \quad (-1)^d \sum_{n=0}^{\infty} I_n(1_{\Delta_n} h_n),$$

where

$$h_n(t_1, \ldots, t_n) = \tag{3}$$
$$\sum_{\substack{k_1 + \cdots + k_d = n \\ k_1 \geq 0, \ldots, k_d \geq 0}} \frac{n!}{k_1! \cdots k_d!} \int_{t^d_{k_d}}^{\infty} \cdots \int_{t^i_{k_i}}^{t^{i+1}_1} \cdots \int_{t^1_{k_1}}^{t^2_1} \partial_1 \cdots \partial_d f(s_1, \ldots, s_d) K^{k_1, \ldots, k_d}_{s_1, \ldots, s_d} ds_1 \cdots ds_d,$$

with, for $0 \leq s_1 \leq \cdots \leq s_d$ and $k_1 \geq 0, \ldots, k_d \geq 0$:

$$K^{k_1, \ldots, k_d}_{s_1, \ldots, s_d} \quad = \quad \sum_{\substack{m_1 \geq n_1, \ldots, m_d \geq n_d \\ m_1 \leq \cdots \leq m_d}} \partial^{k_1} p^{s_1 - s_0}_{m_1 - m_0} \cdots \partial^{k_d} p^{s_d - s_{d-1}}_{m_d - m_{d-1}}, \qquad m_0 = 0, \ s_0 = 0.$$

We make the following remarks.

- The support of the function f can be taken in Δ_d since almost surely, $T_{n_i} \leq T_{n_{i+1}}$, $i = 1, \ldots, d-1$.

- Th. 1 is stated for smooth functions, but by integration by parts it extends easily to square-integrable functionals. For simplicity of notation it is easier in the general case to understand the expression of Th. 1 in distribution sense.

- Th. 1 allows to compute the expansion of a square-integrable random variable which is approximated in $L^2(\Omega)$ by a sequence of functions of finite numbers of Poisson jump times.

Before proving Th. 1 we show how it can be derived in the case $d = 1$. We have the orthogonal expansion

$$1_{\{N_t - N_s = n\}} = \sum_{k \geq 0} \frac{1}{k!(t-s)^k} (1_{\{n\}}, C^{t-s}_k)_{l^2(\mathbb{Z}, p^{t-s})} C^{t-s}_k (N_t - N_s), \quad 0 \leq s \leq t, \ n \in \mathbb{N},$$

hence from (1):

$$1_{\{N_t - N_s = n\}} = \sum_{k\geq 0} \frac{1}{k!} \partial^k p_n^{t-s} I_k(1_{[s,t]}^{\circ k}), \qquad 0 \leq s \leq t, \ n \in \mathbb{N}. \tag{4}$$

From this we obtain for $s = 0$ and $n \geq 1$:

$$1_{[T_n,\infty[}(t) = 1_{\{N_t \geq n\}} = \sum_{k\geq 0} \sum_{l\geq n} \frac{1}{k!} \partial^k p_l^t I_k(1_{[0,t]}^{\circ k}) = \sum_{k\geq 0} \frac{1}{k!} \partial^k P_n(t) I_k(1_{[0,t]}^{\circ k}), \tag{5}$$

where $P_n(t) = \int_0^t p_{n-1}^s ds$ is the distribution function of T_n. We deduce

$$\begin{aligned}
f(T_n) &= -\int_0^\infty f'(s) 1_{[T_n,\infty[}(s) ds \\
&= -\int_0^\infty f'(s) \sum_{k\geq 0} \frac{1}{k!} \partial^k P_n(s) I_k(1_{[0,s]}^{\circ k}) ds \\
&= -\sum_{k\geq 0} \frac{1}{k!} \int_0^\infty f'(s) \partial^k P_n(s) I_k(1_{[0,s]}^{\circ k}) ds \\
&= -\sum_{k\geq 0} \int_0^\infty f'(s) \partial^k P_n(s) \int_0^s \int_0^{t_k} \cdots \int_0^{t_2} d\tilde{N}_{t_1} \cdots d\tilde{N}_{t_k} ds \\
&= -\sum_{k\geq 0} \int_0^\infty f'(s) \partial^k P_n(s) \int_0^\infty \int_0^{t_k} \cdots \int_0^{t_2} 1_{[0,s]}(t_1 \vee \cdots \vee t_k) d\tilde{N}_{t_1} \cdots d\tilde{N}_{t_k} ds \\
&= -\sum_{k\geq 0} \int_0^\infty \int_{t_k}^\infty f'(s) \partial^k P_n(s) ds \int_0^{t_k} \cdots \int_0^{t_2} d\tilde{N}_{t_1} \cdots d\tilde{N}_{t_k},
\end{aligned}$$

hence

$$f(T_n) = -\sum_{k\geq 0} \frac{1}{k!} I_k \left(\int_{t_1\vee\cdots\vee t_k}^\infty f'(s) \partial^k P_n(s) ds \right),$$

which can be rewritten after integration by parts on $\mathbb{R}_+$ as

$$f(T_n) = \sum_{k\geq 0} \frac{1}{k!} I_k \left(f(t_1 \vee \cdots \vee t_k) \partial^{k+1} P_n(t_1 \vee \cdots \vee t_k) + \int_{t_1\vee\cdots\vee t_k}^\infty f(s) \partial^{k+1} P_n(s) ds \right). \tag{6}$$

Proof of Th. 1. We deal with the case $d \geq 2$. Let $0 = s_0 \leq s_1 \leq \cdots \leq s_d$, and $n_1, \ldots, n_d \in \mathbb{N}$. We have from (4) and (2):

$$\begin{aligned}
&\prod_{i=1}^{i=d} 1_{\{N_{s_i} - N_{s_{i-1}} = n_i\}} \\
&= \sum_{n=0}^\infty \sum_{\substack{k_1 + \cdots + k_d = n \\ k_1 \geq 0, \ldots, k_d \geq 0}} \frac{n!}{(k_1! \cdots k_d!)^2} \partial^{k_1} p_{n_1}^{s_1 - s_0} \cdots \partial^{k_d} p_{n_d}^{s_d - s_{d-1}} I_{k_1}(1_{[s_0,s_1]}^{\circ k_1}) \cdots I_{k_d}(1_{[s_{d-1},s_d]}^{\circ k_d}) \\
&= \sum_{n=0}^\infty \sum_{\substack{k_1 + \cdots + k_d = n \\ k_1 \geq 0, \ldots, k_d \geq 0}} \frac{n!}{(k_1! \cdots k_d!)^2} \partial^{k_1} p_{n_1}^{s_1 - s_0} \cdots \partial^{k_d} p_{n_d}^{s_d - s_{d-1}} I_n(1_{[s_0,s_1]}^{\circ k_1} \circ \cdots \circ 1_{[s_{d-1},s_d]}^{\circ k_d}),
\end{aligned}$$

where the last equality used the assumption $s_1 \le \cdots \le s_d$. Now, with $0 \le m_1 \le \cdots \le m_d$,

$$
\begin{aligned}
& 1_{[T_{m_1},T_{m_1+1}[}(s_1) \cdots 1_{[T_{m_d},T_{m_d+1}[}(s_d) = 1_{\{N_{s_1}=m_1\}} \cdots 1_{\{N_{s_d}=m_d\}} \\
&\quad = 1_{\{N_{s_1}-N_{s_0}=m_1-m_0\}} \cdots 1_{\{N_{s_d}-N_{s_{d-1}}=m_d-m_{d-1}\}} \qquad (m_0 = 0) \\
&\quad = \sum_{n=0}^{\infty} \sum_{\substack{k_1+\cdots+k_d=n \\ k_1\ge 0,\ldots,k_d\ge 0}} \frac{n!}{(k_1!\cdots k_d!)^2} \partial^{k_1} p_{m_1-m_0}^{s_1-s_0} \cdots \partial^{k_d} p_{m_d-m_{d-1}}^{s_d-s_{d-1}} I_n(1_{[s_0,s_1]}^{\circ k_1} \circ \cdots \circ 1_{[s_{d-1},s_d]}^{\circ k_d}).
\end{aligned}
$$

Given that $s_1 \le \cdots \le s_d$, for any $i < j$ the conditions $s_i \in [T_{m_i}, T_{m_{i+1}}[$ and $s_j \in [T_{m_j}, T_{m_{j+1}}[$ imply $m_i \le m_j$, hence

$$
\begin{aligned}
\prod_{i=1}^{i=d} 1_{[T_{n_i},\infty[}(s_i) &= \sum_{\substack{m_1\ge n_1,\ldots,m_d\ge n_d \\ m_1\le\cdots\le m_d}} 1_{[T_{m_1},T_{m_1+1}[}(s_1) \cdots 1_{[T_{m_d},T_{m_d+1}[}(s_d) \\
&= \sum_{\substack{m_1\ge n_1,\ldots,m_d\ge n_d \\ m_1\le\cdots\le m_d}} 1_{\{N_{s_1}=m_1\}} \cdots 1_{\{N_{s_d}=m_d\}} \\
&= \sum_{\substack{m_1\ge n_1,\ldots,m_d\ge n_d \\ m_1\le\cdots\le m_d}} 1_{\{N_{s_1}-N_{s_0}=m_1-m_0\}} \cdots 1_{\{N_{s_d}-N_{s_{d-1}}=m_d-m_{d-1}\}} \\
&= \sum_{n=0}^{\infty} \sum_{\substack{k_1+\cdots+k_d=n \\ k_1\ge 0,\ldots,k_d\ge 0}} \frac{n!}{(k_1!\cdots k_d!)^2} \\
&\qquad \sum_{\substack{m_1\ge n_1,\ldots,m_d\ge n_d \\ m_1\le\cdots\le m_d}} \partial^{k_1} p_{m_1-m_0}^{s_1-s_0} \cdots \partial^{k_d} p_{m_d-m_{d-1}}^{s_d-s_{d-1}} I_n(1_{[s_0,s_1]}^{\circ k_1} \circ \cdots \circ 1_{[s_{d-1},s_d]}^{\circ k_d}) \\
&= \sum_{n=0}^{\infty} \sum_{\substack{k_1+\cdots+k_d=n \\ k_1\ge 0,\ldots,k_d\ge 0}} \frac{n!}{(k_1!\cdots k_d!)^2} K_{s_1,\ldots,s_d}^{k_1,\ldots,k_d} I_n(1_{[s_0,s_1]}^{\circ k_1} \circ \cdots \circ 1_{[s_{d-1},s_d]}^{\circ k_d}).
\end{aligned}
$$

Using the identity

$$
\begin{aligned}
& f(T_{n_1}, \ldots, T_{n_d}) \\
&\quad = (-1)^d \int_0^{\infty} \cdots \int_0^{\infty} 1_{[T_{n_1},\infty[}(s_1) \cdots 1_{[T_{n_d},\infty[}(s_d) \partial_1 \cdots \partial_d f(s_1, \ldots, s_d) ds_1 \cdots ds_d \\
&\quad = (-1)^d \int_{\Delta_d} 1_{[T_{n_1},\infty[}(s_1) \cdots 1_{[T_{n_d},\infty[}(s_d) \partial_1 \cdots \partial_d f(s_1, \ldots, s_d) ds_1 \cdots ds_d, \quad f \in \mathcal{C}_c^d(\Delta_d),
\end{aligned}
$$

we get

$$
\begin{aligned}
f(T_{n_1}, \ldots, T_{n_d}) = (-1)^d \sum_{n=0}^{\infty} \sum_{\substack{k_1+\cdots+k_d=n \\ k_1\ge 0,\ldots,k_d\ge 0}} \frac{n!}{(k_1!\cdots k_d!)^2} \\
\int_{\Delta_d} \partial_1 \cdots \partial_d f(s_1, \ldots, s_d) K_{s_1,\ldots,s_d}^{k_1,\ldots,k_d} I_n(1_{[s_0,s_1]}^{\circ k_1} \circ \cdots \circ 1_{[s_{d-1},s_d]}^{\circ k_d}) ds_1 \cdots ds_d.
\end{aligned}
$$

From (2), we have for $s_1 \leq \cdots \leq s_d$ and $k_1 \geq 0, \ldots, k_d \geq 0$:

$$\begin{aligned}
& I_n \left(1_{[s_0,s_1]}^{\circ k_1} \circ \cdots \circ 1_{[s_{d-1},s_d]}^{\circ k_d} \right) \\
&= k_1! \cdots k_d! \int_0^\infty \int_0^{t^d_{k_d}} \cdots \int_0^{t^1_2} 1_{[s_0,s_1]}^{\circ 2}(t^1_1, t^1_{k_1}) \cdots 1_{[s_{d-1},s_d]}^{\circ 2}(t^d_1, t^d_{k_d}) d\tilde{N}_{t^1_1} \cdots d\tilde{N}_{t^d_{k_d}},
\end{aligned}$$

hence exchanging the stochastic integrals and the integrals with respect to $ds_1 \cdots ds_d$, we obtain

$$f(T_{n_1}, \ldots, T_{n_d}) = (-1)^d \sum_{n=0}^{\infty} \sum_{\substack{k_1 + \cdots + k_d = n \\ k_1 \geq 0, \ldots, k_d \geq 0}} \frac{n!}{k_1! \cdots k_d!}$$

$$I_n \left(1_{\Delta_n} \int_{t^d_{k_d}}^{\infty} \int_{t^{d-1}_{k_{d-1}}}^{t^d_1} \cdots \int_{t^2_{k_2}}^{t^3_1} \int_{t^1_{k_1}}^{t^2_1} \partial_1 \cdots \partial_d f(s_1, \ldots, s_d) K^{k_1,\ldots,k_d}_{s_1,\ldots,s_d} ds_1 \cdots ds_d \right). \qquad \square$$

Th. 1 can be compared to its Gaussian counterpart in the following way. For $u \in L^2(\mathbb{R}_+)$, let $J_1(u)$ denote the Itô-Wiener integral $J_1(u) = \int_0^\infty u(s) dB_s$ with respect to the Brownian motion $(B_s)_{s \in \mathbb{R}_+}$ defined on the Wiener space W. Then,

$$\begin{aligned}
& f(J_1(u_1), \ldots, J_1(u_d)) \qquad (7) \\
&= \sum_{n=0}^{\infty} \sum_{\substack{k_1 + \cdots + k_d = n \\ k_1 \geq 0, \ldots, k_d \geq 0}} \frac{n!}{(k_1! \cdots k_d!)^2} (f, \partial_1^{k_1} \cdots \partial_d^{k_d} p)_{L^2(\mathbb{R}^d)} J_n(u_1^{\circ k_1} \circ \cdots \circ u_d^{\circ k_d}),
\end{aligned}$$

where $\{u_1, \ldots, u_d\}$ is an orthonormal family in $L^2(\mathbb{R}_+)$ and $p(s_1, \ldots, s_d)$ is the standard Gaussian density function of the vector $(J_1(u_1), \ldots, J_1(u_d))$. The multiple Wiener integral

$$J_n(u_1^{\circ k_1} \circ \cdots \circ u_d^{\circ k_d}) = H_{k_1}(J_1(u_1)) \cdots H_{k_d}(J_1(u_d)) \qquad (8)$$

$n = k_1 + \cdots + k_d$, being defined with help of the Hermite polynomials $(H_m)_{m \geq 0}$ with

$$H_{k_1}(s_1) \cdots H_{k_d}(s_d) = \frac{(-1)^{k_1 + \cdots + k_d}}{p(s_1, \ldots s_d)} \partial_{s_1}^{k_1} \cdots \partial_{s_d}^{k_d} p(s_1, \ldots, s_d).$$

4 A multiplicative exponential on Poisson space

Given the role played by the iterated derivatives of the Poisson and Gaussian laws in the chaos expansions of $f(T_1, \ldots, T_d)$ and $f(I_1(u_1), \ldots, I_d(u_d))$, it is natural to determine all random variables whose development is determined in such a way by their density function. For this we will define exponential vectors on Fock space that differ from Wick exponentials. For $g \in \mathcal{S}(\mathbb{R})$ and $t \in \mathbb{R}_+$ such that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mid \partial^n G(t) \mid^2 < \infty,$$

let $\mathcal{E}_g(t)$ be defined in $L^2(\Omega)$ by

$$\mathcal{E}_g(t) = \sum_{n\geq 0} \frac{1}{n!} \partial^n G(t) I_n(1_{[0,t]}^{\circ n}),$$

where $G(t) = \int_0^t g(s)ds$, $t \in \mathbb{R}_+$. From the expression of $I_n(1_{[0,t]}^{\circ n})$ in terms of Charlier polynomials, $\mathcal{E}_g(t)$ is of the form $\mathcal{E}_g(t) = h(N_t, t)$. For $f \in \mathcal{S}(\mathbb{R})$, let

$$\mathcal{E}_g(f) = -\int_0^\infty f'(s)\mathcal{E}_g(s)ds.$$

Since $\mathcal{E}_{p_k}(t)$ is identified to $1_{[T_{k+1},\infty[}(t)$, from (5) this means that $\mathcal{E}_{p_k}(f) = f(T_{k+1})$, $k \in \mathbb{N}$. The following lemma gives the product rule for $\mathcal{E}_g(t)$. We note that $D\mathcal{E}_g(t) = 1_{[0,t]}\mathcal{E}_{\partial g}(t)$. We use the convention $\partial^{-1} f(t) = \int_0^t f(s)ds$.

Lemma 1 *Let $t \in \mathbb{R}_+$. Let $f, g \in \mathcal{S}(\mathbb{R})$ such that for any $t \in \mathbb{R}_+$ there exists $A_t \geq 1$, with $\mid \partial^i f(t) \mid \leq (A_t)^{i+1}$ and $\mid \partial^i g(t) \mid \leq (A_t)^{i+1}$, $i \geq -1$. Then $\mathcal{E}_f(t)\mathcal{E}_g(t) \in L^2(\Omega)$ and its chaos expansion is given by*

$$\mathcal{E}_f(t)\mathcal{E}_g(t) = \mathcal{E}_h(t),$$

where the function h is defined as

$$h(s) = \frac{d}{ds}(\mathcal{E}_f(s), \mathcal{E}_g(s))_{L^2(\Omega)}, \quad s \in \mathbb{R}_+.$$

Proof. We use the formula $F = E[F] + \sum_{n\geq 1} \frac{1}{n!} I_n(E[D^n F])$ of [8], cf. also [1], and an induction argument on n to show that

$$E[D^n(\mathcal{E}_f(t)\mathcal{E}_g(t))] = \left(1_{[0,t]}\right)^{\circ n} \partial^n \left(\sum_{i\in\mathbb{N}} \frac{t^i}{i!} \partial^{i-1} f(t) \partial^{i-1} g(t) \right), \quad n \in \mathbb{N}, \ t \in \mathbb{R}_+.$$

The result is clear for $n = 0$. Since D is a finite difference operator, we have by induction

$$\begin{aligned}
&E[D^{n+1}(\mathcal{E}_f(t)\mathcal{E}_g(t))] \\
&= E[D^n(\mathcal{E}_f(t)D\mathcal{E}_g(t) + \mathcal{E}_g(t)D\mathcal{E}_f(t) + D\mathcal{E}_f(t)D\mathcal{E}_g(t))] \\
&= 1_{[0,t]}E[D^n(\mathcal{E}_f(t)\mathcal{E}_{\partial g}(t) + \mathcal{E}_g(t)\mathcal{E}_{\partial f}(t) + \mathcal{E}_{\partial f}(t)\mathcal{E}_{\partial g}(t))] \\
&= 1_{[0,t]}^{\circ(n+1)} \partial^n \left(\sum_{i\geq 0} \frac{t^i}{i!} \partial^{i-1} f(t) \partial^i g(t) + \sum_{i\geq 0} \frac{t^i}{i!} \partial^{i-1} g(t) \partial^i f(t) + \sum_{i\geq 0} \frac{t^i}{i!} \partial^i g(t) \partial^i f(t) \right) \\
&= 1_{[0,t]}^{\circ(n+1)} \partial^{n+1} \sum_{i\geq 0} \frac{t^i}{i!} \partial^{i-1} g(t) \partial^{i-1} f(t), \quad n \geq 0.
\end{aligned}$$

Moreover, the growth conditions imposed on the derivatives of f et g imply that

$$
\begin{aligned}
\mid \partial^n h(t) \mid & \leq \sum_{i\geq 0} \frac{t^i}{i!} \sum_{j=0}^{j=n+1} \left(\begin{array}{c} n+1 \\ j \end{array} \right) \mid \partial^{i+j-1} g(t) \partial^{i+n-j} f(t) \mid \\
& \leq 2^{n+1} \sum_{i\geq 0} \frac{t^i}{i!} (A_t)^{2i+n+1} \leq 2^{n+1} (A_t)^{n+1} e^{t(A_t)^2},
\end{aligned}
$$

hence

$$
E[\mathcal{E}_h(t)^2] = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mid \partial^{n-1} h(t) \mid^2 \leq \exp\left(6t(A_t)^2\right) < \infty, \qquad t \in \mathbb{R}_+. \qquad \square
$$

Note that p_n^t satisfies the hypothesis of the above Lemma, with $A_t = 2(t \vee 1)$, $t \in \mathbb{R}_+$. The following result shows that $\mathcal{E}_{p_\alpha}(t)$ has a Bernoulli distribution with parameter $P_\alpha(t)$, $t \in \mathbb{R}_+$ only for integer values of α, and gives a probabilistic solution of a differential equation.

Proposition 1 *Let $g \in \mathcal{S}(\mathbb{R})$, with $G(t) = \int_0^t g(s)ds$, $t \in \mathbb{R}_+$, and such that for any $t \in \mathbb{R}_+$ there exists $A_t > 1$, with $\mid \partial^i f(t) \mid \leq (A_t)^{i+1}$, $i \geq -1$. The following statements are equivalent.*
(i) The function g is written as

$$
g = -\sum_{k\in\mathbb{N}} \alpha_k \partial p_k = \alpha_0 p_0 + \sum_{k\geq 1} \alpha_k (p_k - p_{k-1}), \quad (\alpha_n)_{n\in\mathbb{N}} \subset \{0,1\},
$$

(ii) The random variable $\mathcal{E}_g(t)$ is an indicator function, $\forall t \in \mathbb{R}_+$,
(iii) G solves the nonlinear equation

$$
G(t) = \sum_{n\geq 0} \frac{t^n}{n!} (\partial^n G(t))^2, \ t \in \mathbb{R}_+,
$$

Proof. The implication $(i) \Rightarrow (ii)$ is follows from the identity

$$
\mathcal{E}_g(t) = \sum_{k\in\mathbb{N}} \alpha_k 1_{\{N_t = k\}},
$$

cf. (4). If (ii) is satisfied, then $\mathcal{E}_g(t)\mathcal{E}_g(t) = \mathcal{E}_g(t)$, hence from Lemma 1 (iii) holds. Conversely, (iii) can be stated as $G(t) = E[\mathcal{E}_g(t)\mathcal{E}_g(t)]$, which implies $\mathcal{E}_g(t)\mathcal{E}_g(t) = \mathcal{E}_g(t)$ from Lemma 1, i.e. $\mathcal{E}_g(t)$ is an indicator function. This proves $(ii) \Leftrightarrow (iii)$. If (ii) is satisfied, then $\mathcal{E}_g(t)$ is of the form

$$
\mathcal{E}_g(t) = \sum_{n\in\mathbb{N}} \alpha_n(t) 1_{\{N_t = n\}},
$$

with $\alpha_n(t) \in \{0,1\}$, $n \in \mathbb{N}$, $t \in \mathbb{R}_+$, and

$$\mathcal{E}_g(t) = \alpha_0(t)(1 - \mathcal{E}_{p_1}(t)) + \sum_{n\geq 1} \alpha_n(t)(\mathcal{E}_{p_n}(t) - \mathcal{E}_{p_{n-1}}(t)).$$

Identifying the first chaos terms we have

$$g(t) = \alpha_0(t)p_0(t) + \sum_{n\geq 1} \alpha_n(t)(p_n(t) - p_{n-1}(t)).$$

Since g is continuous, α_n is continuous in t, hence constant, which implies (i).

$\square$

5 The Gaussian case

In this section we compare the notions introduced above with their analogs on the Wiener space. For $u \in L^2(\mathbb{R}_+)$ with $\| u \|_{L^2(\mathbb{R}_+)} = 1$, and $g \in \mathcal{S}(\mathbb{R})$ such that

$$\sum_{k=0}^{\infty} \frac{1}{k!} \mid \partial^k G(t) \mid^2 < \infty,$$

let

$$\mathcal{E}_g(t) = \sum_{k\geq 0} \frac{\partial^k G(t)}{k!} J_k(u^{\circ k}),$$

where $J_k(u)$ is the multiple Wiener integral defined in (8). If g is a Gaussian density function with variance 1 and mean a, then from (7),

$$\mathcal{E}_g(t) = 1_{]-\infty, t-a]}(J_1(u)).$$

The product rule for $\mathcal{E}_g(t)$ on the Wiener space is the same as on the Poisson space.

Lemma 2 *Let $t \in \mathbb{R}_+$ and $f, g \in \mathcal{S}(\mathbb{R})$ such that for any $t \in \mathbb{R}_+$ there exists $A_t \geq 1$, with $\mid \partial^i f(t) \mid \leq (A_t)^{i+1}$ and $\mid \partial^i g(t) \mid \leq (A_t)^{i+1}$, $i \geq -1$. Then $\mathcal{E}_f(t)\mathcal{E}_g(t) \in L^2(\Omega)$ and its chaos expansion is given by*

$$\mathcal{E}_f(t)\mathcal{E}_g(t) = \mathcal{E}_h(t),$$

where the function h is defined as

$$h(s) = \frac{d}{ds}(\mathcal{E}_f(s), \mathcal{E}_g(s))_{L^2(W)}, \quad s \in \mathbb{R}_+.$$

Proof. The the formula $F = E[F] + \sum_{n\geq 1}\frac{1}{n!}J_n(E[D^n F])$ is still valid on Wiener space and as in Lemma. 1 we show by induction that

$$E[D^n(\mathcal{E}_f(t)\mathcal{E}_g(t))] = \left(1_{[0,t]}\right)^{\circ n}\partial^n\left(\sum_{i\in\mathbb{N}}\frac{1}{i!}\partial^{i-1}f(t)\partial^{i-1}g(t)\right), \quad n\in\mathbb{N}.$$

Since D is identified to a derivation operator, cf. [4], we have

$$\begin{aligned}
E[D^{n+1}(\mathcal{E}_f(t)\mathcal{E}_g(t))] &= E[D^n(\mathcal{E}_f(t)D\mathcal{E}_g(t)+\mathcal{E}_g(t)D\mathcal{E}_f(t))] \\
&= 1_{[0,t]}E[D^n(\mathcal{E}_f(t)\mathcal{E}_{\partial g}(t)+\mathcal{E}_g(t)\mathcal{E}_{\partial f}(t))] \\
&= 1_{[0,t]}^{\circ(n+1)}\partial^n\left(\sum_{i\geq 0}\frac{1}{i!}\partial^{i-1}f(t)\partial^i g(t)+\sum_{i\geq 0}\frac{1}{i!}\partial^{i-1}g(t)\partial^i f(t)\right) \\
&= 1_{[0,t]}^{\circ(n+1)}\partial^{n+1}\sum_{i\geq 0}\frac{1}{i!}\partial^{i-1}g(t)\partial^{i-1}f(t), \quad n\geq 0.
\end{aligned}$$

It follows as in the proof of Lemma 1 that $\mathcal{E}_f(t)\mathcal{E}_g(t)$ belongs to $L^2(\Omega)$.

□

Proposition 2 *Let $g\in\mathcal{S}(\mathbb{R})$, with $G(t)=\int_0^t g(s)ds$, $t\in\mathbb{R}_+$, such that for any $t\in\mathbb{R}_+$ there exists $A_t\geq 1$, with $\mid\partial^i f(t)\mid\leq (A_t)^{i+1}$ and $\mid\partial^i g(t)\mid\leq (A_t)^{i+1}$, $i\geq -1$. The following statements are equivalent.*
(i) The random variable $\mathcal{E}_g(t)$ is an indicator function, $\forall t\in\mathbb{R}_+$,
(ii) G solves the nonlinear equation

$$G(t)=\sum_{n\geq 0}\frac{1}{n!}(\partial^n G(t))^2,\ t\in\mathbb{R}_+.$$

Moreover, these statements hold if
(iii) the function g is a Gaussian density function with variance one.

Proof. The implication $(iii)\Rightarrow(i)$ holds because in this case, $\mathcal{E}_g(t)=1_{]-\infty,a(t)]}(J_1(u))$ for some $a(t)\in\mathbb{R}$, and $(i)\Leftrightarrow(ii)$ follows from Lemma 2.

□

The exponential vector $e^{\frac{1}{2}\|u\|^2}\varepsilon(u)$ on Wiener space is obtained as

$$e^{\frac{1}{2}\|u\|^2}\varepsilon(u)=-\int_0^\infty e^s\mathcal{E}_p(s)ds=e^{J_1(u)},$$

and satisfies the product identity

$$\varepsilon(u)\varepsilon(v)=\varepsilon(u+v)\exp((u,v)_{L^2(\mathbb{R}_+)}).$$

On the Poisson space, however, this multiplicative property disappears because the Wick exponential is interpreted as a discrete product given as a solution of a stochastic differential equation. A family of exponential vectors with multiplicative property can be defined on Poisson space as

$$\tilde{\varepsilon}_k(u) = -\int_0^\infty u'(s)e^{u(s)}\mathcal{E}_{p_k}(s)ds = e^{u(T_k)}, \quad k \geq 1.$$

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