# An extension of stochastic calculus to certain non-Markovian processes

## Nicolas Privault

*Equipe d'Analyse et Probabilités, Université d'Evry-Val d'Essonne*
*Boulevard des Coquibus, 91025 Evry Cedex, France*[1]
*e-mail:* `privault@lami.univ-evry.fr`

**Abstract**

By time changes of Lévy processes we construct two operators on Fock space whose sum is a second quantized operator, and that complement the annihilation and creation operators whose probabilistic interpretations use shifts of trajectories. This results in an analytic construction, for certain non-Markovian processes, of stochastic calculus including Itô differentials, generators and associated integro-differential equations, without using the notion of filtration.

**Key words:** Anticipating stochastic calculus, Malliavin calculus, Fock space, Lévy processes.

*Mathematics Subject Classification (1991):* 60G55, 60H07, 60G57, 60G50

# 1 Introduction

The stochastic calculus of variations on the Wiener space, cf. [12], allows to construct an anticipating stochastic calculus for Brownian motion via the Skorohod integral, cf. e.g. [14], [15]. Extensions of this anticipating stochastic calculus in the jump case have been considered in [6], [16], [19], however they only concern the Poisson process on the real line, or time-changed Poisson processes, cf. [20]. The regularity of laws of solutions of stochastic differential equations of jump type have been studied in [4], [5], [18]. There are well-known links between Fock space and stochastic calculus that usually involve the annihilation and creation operators and their probabilistic interpretation by shifts of trajectories of the Wiener and Poisson processes. In this paper we introduce two operators on Fock space that are interpreted probabilistically by time changes of the Wiener and Poisson processes. We establish a formula that

---

[1]Present address: Centre de Recerca Matemàtica, Apartat 50, 08193 Bellaterra, Spain.

expresses on random variables the infinitesimal action of time changes performed on a Lévy process $\tilde{X}(dx, dt)$ with Brownian component $B(dt)$ and compensated Poisson component $\tilde{N}(dx, dt)$ on $M \times \mathbb{R}_+$, and study its connections to the Itô formula. For a sufficiently smooth random variable $F$,

$$-\frac{d}{d\varepsilon} \mathcal{T}_{\varepsilon h} F_{|\varepsilon=0} = a_h^\ominus F + \int_{M \times \mathbb{R}_+} h(x, t) \left( 1_{\{x \neq 0\}} \nabla_{x,t}^- + \frac{1}{2} 1_{\{x=0\}} \nabla_{0,t}^- \nabla_{0,t}^- \right) F \mu(dx) dt.$$

The operator $\nabla^\ominus$ is defined by means of chaotic decompositions, $\nabla^-$ is the annihilation operator on Fock space, and $\mathcal{T}_h F$ is defined by evaluationg $F$ on time changed trajectories of the Lévy process. (This time change is governed by the function $h \in L^\infty(\mathbb{R}_+)$). The term $a_h^\ominus F$ has expectation is zero and is interpreted as a martingale term. The sum of the operator $\nabla^\ominus$ and its adjoint $\nabla^\oplus$ gives a second quantized operator on Fock space close to the number operator. This can be interpreted as a decomposition of the number operator process (or Poisson process in quantum probability) into creation and annihilation parts, by analogy with the well-known decomposition of Gaussian white noise. With help of the operator

$$\mathcal{A}_t = \int_M \left( 1_{\{x \neq 0\}} \nabla_{x,t}^- + \frac{1}{2} 1_{\{x=0\}} \nabla_{0,t}^- \nabla_{0,t}^- \right) d\mu(x), \quad t \in \mathbb{R}_+,$$

we associate a notion of generator and a class of partial differential equations that can have negative second order coefficient to processes that anticipate the Lévy filtration, or are not Markovian. The "Wiener part" $\frac{1}{2} \nabla_{0,t}^- \nabla_{0,t}^-$ of $\mathcal{A}_t$ is identical, after integration with respect to $dt$, to the Gross Laplacian on Wiener space, cf. [9], [10]. The Itô formula is written for non-Markovian processes and we obtain the chaos expression

$$\nabla_t^\ominus f(X_{t^+}^h) = f'(X_t^h) \partial_t B(t) + \int_M (f(X_t^h + h(x, t)) - f(X_t^h)) \partial_t \tilde{N}(dx, t), \quad t \in \mathbb{R}_+,$$

of the martingale term in the Itô formula, where $(X_t^h)_{t \in \mathbb{R}_+}$ is the uncompensated process $X_t^h = \int_M \int_0^t h(x, s) X(dx, ds)$. (See Prop. 9 for a precise version of this statement). As for many results in anticipating stochastic calculus, the extensions are obtained provided some regularity assumptions are made on the stochastic processes.

Another goal of this paper is to construct an anticipating stochastic calculus for Lévy processes. As in the standard Poisson case, cf. [6], [19], we obtain for the jump part of the Lévy process $X$ two different notions of gradient and Skorohod integral, depending on the type (space or time) of perturbation chosen.

The organization of this paper is as follows. Sect. 2 contains preliminaries

on Fock space and Lévy processes. We recall the construction of infinitely divisible random variables as operators on Fock space and their application to the representation of Lévy processes as operator processes. This construction can be found in the work of quantum probabilists, cf. [2], [3], [17]. Our proof uses the Itô formula for multiple Poisson-Wiener integrals instead of the quantum probabilistic argument which is based on the Weyl representation and commutation relations. In Sect. 3 the operators $\nabla^\ominus$, $a_h^\ominus$ and their adjoints are defined. Sect. 4 presents the different ways to perturb the Lévy process by perturbation of space or time, and gives the interpretation of these perturbations with $\nabla^-$, $\nabla^\ominus$ in terms of Fock space. In Sect. 5 we state the extension of the Itô formula and give the chaos expansion of its "martingale term". In the general form of this result, some smoothness must be imposed on the considered functionals, i.e. the formula may hold only in distribution sense. In Sect. 6 we study the connection between our form of the Itô formula and its associated integro-differential equations, with different examples. We also discuss possible directions for the extension of our construction. In Sect. 7 two different Skorohod integrals are constructed depending on the type of perturbation chosen (times changes or shifts of trajectories). Their properties as extensions of the stochastic integral are stated given a natural definition of adaptedness, unifying the different notions of gradient and Skorohod integral on the Wiener and Poisson spaces, cf. [6], [8], [16]. Sect. 8 deals with the Clark formula for Lévy processes.

# 2   Representation of Lévy processes as operator processes on Fock space

This section consists essentially in definitions. Its main result is Prop. 1 which gives the action on the Fock space of the multiplication operator by a single stochastic integral.

## 2.1   Creation and annihilation on Fock space

The Fock space $\Gamma(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is defined as the direct sum

$$\Gamma(\mathcal{H}) = \bigoplus_{n \geq 0} \mathcal{H}^{\circ n},$$

where the symmetric tensor product $\mathcal{H}^{\circ n}$ is endowed with the norm

$$\| \cdot \|_{\mathcal{H}^{\circ n}}^2 = n! \, \| \cdot \|_{\mathcal{H}^{\otimes n}}^2, \quad n \in \mathbf{N}.$$

Here, "⊗" denotes the completed tensor product and "∘" denotes its symmetrization. The annihilation and creation operators $\nabla^- : \Gamma(\mathcal{H}) \longrightarrow \Gamma(\mathcal{H}) \otimes \mathcal{H}$ and $\nabla^+ : \Gamma(\mathcal{H}) \otimes \mathcal{H} \longrightarrow \Gamma(\mathcal{H})$ are densely defined as

$$\nabla^- h^{\circ n} = n h^{\circ n-1} \otimes h, \qquad \nabla^+ \left( h^{\circ n} \otimes g \right) = h^{\circ n} \circ g, \quad n \in \mathbb{N}, \tag{1}$$

by linearity and polarization. They are closable, of domains $Dom(\nabla^-)$ and $Dom(\nabla^+)$. Let $\mathcal{E}$ denote the dense set of elements of $\Gamma(\mathcal{H})$ that have a finite chaotic development. Given $h \in \mathcal{H}$, the closable operators $a_h^-$, $a_h^+$ of quantum probability, cf. [13], [17], are densely defined as

$$a_h^- F = (\nabla^- F, h)_{\mathcal{H}}, \quad a_h^+ F = \nabla^+ (F \otimes h), \quad F \in \mathcal{E}. \tag{2}$$

If $A$ is an operator on $\mathcal{H}$, the differential second quantization of $A$ is the linear operator $d\Gamma(A)$ defined by

$$d\Gamma(A) (h_1 \circ \cdots \circ h_n) = \sum_{k=1}^{k=n} h_1 \circ \cdots \circ A h_i \circ \cdots \circ h_n,$$

$h_1, \ldots, h_n \in Dom(A)$, i.e. $d\Gamma(A)F = \nabla^+ (A\nabla^- F)$, and $d\Gamma(I_d)$ is the number operator.

## 2.2 Lévy processes and stochastic integrals

Let $M$ be a metric space with Borel $\sigma$-algebra $\mathcal{M}$. We only assume that $M$ contains an element denoted by 0, with $\mid x \mid = d(x, 0)$, so that $M$ can be a manifold, and let $M^* = M \setminus \{0\}$. Let $\mu$ be a $\sigma$-finite Radon measure on $(M, \mathcal{M})$ such that

$$\int_M \mid x \mid^2 \wedge 1 \mu(dx) < \infty,$$

with $\mu(\{0\}) = 1$. Consider a Lévy process of the form

$$\tilde{X}(dx, dt) = X(dx, dt) - \mu(dx)dt = dB_t + N(dx, dt) - \mu(dx)dt,$$

where $\tilde{N}(dx, dt) = N(dx, dt) - \mu(dx)dt$ is a compensated Poisson random measure on $M^* \times \mathbb{R}_+$ of intensity $\mu(dx)dt$, and $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion independent of $N(dx, ds)$. The underlying probability space is denoted by $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is generated by $X$. As a convention we set $L^2(M) = L^2(M, \mu)$, $L^2(M \times \mathbb{R}_+) = L^2(M \times \mathbb{R}_+, \mu(dx)dt)$, and $L^2(\Omega) = L^2(\Omega, P)$. We define the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ as $\mathcal{F}_t = \sigma(X(dx, ds) : x \in M, s \leq t)$. A process $u \in L^2(\Omega) \otimes L^2(M \times \mathbb{R}_+)$

is said to be $\mathcal{F}_t$-adapted if $(\int_M h(x)u(x,t)d\mu(x))_{t\in\mathbb{R}_+}$ is adapted for any $h \in \mathcal{C}_c(M)$. The integral of a square-integrable $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted process $u \in L^2(\Omega) \otimes L^2(M \times \mathbb{R}_+)$ with respect to $\tilde{X}(dx,dt)$ is written as

$$\int_{M\times\mathbb{R}_+} u(x,t)\tilde{X}(dx,dt),$$

with the isometry

$$E\left[\left(\int_{M\times\mathbb{R}_+} u(x,t)\tilde{X}(dx,dt)\right)^2\right] = E\left[\int_{M\times\mathbb{R}_+} u^2(x,t)\mu(dx)dt\right], \qquad (3)$$

and the multiple stochastic integral $I_n(h_n)$ of $h_n \in L^2(M \times \mathbb{R}_+)^{\circ n}$ can be defined by induction with

$$I_n(h_n) = n \int_{M\times\mathbb{R}_+} I_{n-1}(\pi_{x,t}^n h_n)\tilde{X}(dx,dt),$$

where

$$\pi_{x,t}^n : L^2(M \times \mathbb{R}_+)^{\circ n} \longrightarrow L^2(M \times \mathbb{R}_+)^{\circ(n-1)} \qquad (4)$$

is defined by

$$\left[\pi_{x,t}^n h_n\right](x_1,t_1,\ldots,x_{n-1},t_{n-1}) = h_n(x_1,t_1,\ldots,x_{n-1},t_{n-1},x,t)1_{[0,t]}(t_1)\cdots 1_{[0,t]}(t_{n-1}),$$

for $x_1,\ldots,x_{n-1},x \in M$ and $t_1,\ldots,t_{n-1},t \in \mathbb{R}_+$. The isometry property

$$E\left[I_n(h_n)^2\right] = n! \parallel h_n \parallel_{L^2(M\times\mathbb{R}_+)^{\otimes n}}^2,$$

follows from (3). Let $h \in L^2(M \times \mathbb{R}_+)$. The characteristic function of

$$I_1(h) = \int_{M\times\mathbb{R}_+} h(x,t)\tilde{X}(dx,dt) = \int_{M^*\times\mathbb{R}_+} h(x,t)\tilde{N}(dx,dt) + \int_0^\infty h(0,t)dB_t$$

is given by the Lévy-Khintchine formula

$$E\left[e^{izI_1(h)}\right] = \exp\left(-\frac{1}{2}z^2\int_0^\infty h(0,t)^2 dt + \int_{M^*\times\mathbb{R}_+}(e^{izh(x,t)} - 1 - izh(x,t))\mu(dx)dt\right).$$

## 2.3   Chaotic calculus

In the remaining of this paper we work on the Fock space $\Gamma(\mathcal{H})$, with $\mathcal{H} = L^2(M \times \mathbb{R}_+)$, and let $\mathcal{K} = L^2(M)$. Let $\mathcal{C} = \mathcal{C}_c^1(\mathbb{R}_+,\mathcal{C}_c(M)) \cap \mathcal{C}_c(M \times \mathbb{R}_+)$.

**Definition 1** *Let $\mathcal{S}$ denote the vector subspace of $\Gamma(\mathcal{H})$ generated by elements of the form $h_1 \circ \cdots \circ h_n$, where $h_1,\ldots,h_n \in \mathcal{C}$, $n \in \mathbb{N}$.*

Elements of $\Gamma(\mathcal{H})$ are identified with random variables in $L^2(\Omega)$, by associating $h_n \in L^2(M \times \mathbb{R}_+)^{\circ n}$ to its multiple stochastic integral $I_n(h_n)$, building the classical linear isometric isomorphism from $\Gamma(\mathcal{H})$ onto $L^2(\Omega)$. For $F \in \mathcal{S}$, $\nabla F \in \Gamma(\mathcal{H}) \otimes L^2(M \times \mathbb{R}_+)$ is identified to a square-integrable function on $M \times \mathbb{R}_+$ with values in $\mathcal{S}$, and this function will be denoted as $(\nabla^-_{x,t} F)_{(x,t) \in M \times \mathbf{R}_+}$. In Sect. 4, elements of $\mathcal{S}$ will be interpreted via the Fock space isomorphism as smooth random variables that will be defined everywhere, that is for every trajectory of the Lévy process $X$.

We denote by $\pi : L^2(M \times \mathbb{R}_+) \to L^2(M^* \times \mathbb{R}_+)$ the canonical projection. Let $h \in L^2(M \times \mathbb{R}_+)$, let $H$ denote the multiplication operator by the function $h$, and let $\hat{h} = \pi h$. We define the operator $Y^h$ on the dense domain $\mathcal{S}$ in $\Gamma(\mathcal{H})$ as:

$$Y^h F = \nabla^+ \left( H\pi\nabla^- F \right) + \nabla^+ (F \otimes h) + (\nabla^- F, h)_{\mathcal{H}}.$$

The following is an adaptation of a result of [17] with a different proof.

**Proposition 1** *The operator $Y^h = d\Gamma(H\pi) + a^-_h + a^+_h$ on $\Gamma(\mathcal{H})$ acts by multiplication by $I_1(h)$ on $L^2(\Omega)$ under the identification between $\Gamma(\mathcal{H})$ and $L^2(\Omega)$.*

*Proof.* We use the Itô formula for multiple Wiener-Poisson $n$-th and first order stochastic integrals, cf. e.g. [23], [24]. Let $g \in \mathcal{C}$. We have

$$I_1(h)I_n(g^{\circ n}) = nI_n(g^{\circ(n-1)} \circ (\hat{h}g)) + I_{n+1}(g^{\circ n} \circ h) + n(g, h)_{\mathcal{H}} I_{n-1}(g^{\circ(n-1)}), \qquad (5)$$

$n \geq 1$, and this identity can be rewritten as

$$I_1(h)I_n(g^{\circ n}) \quad = \quad \nabla^+(H\pi\nabla^- I_n(g^{\circ n})) + \nabla^+(I_n(g^{\circ n}) \otimes h) + (\nabla^- I_n(g^{\circ n}), h)_{\mathcal{H}}. \quad \square$$

This proposition will be used in Sect. 7 for the construction of the Skorohod integral. From (5) we have $\mathcal{S} \subset \cap_{p \geq 2} L^p(\Omega)$. If $h \in L^\infty(M \times \mathbb{R}_+)$ has finite measure support, Prop. 1 gives

$$\left(\Phi, \exp(-izY^h)\Phi\right)_{\Gamma(L^2(\mathcal{H}))}$$
$$= \quad \exp\left(-\frac{1}{2}z^2 \int_0^\infty h^2(0, s)ds + \int_{M^* \times \mathbf{R}_+} (e^{izh(x,s)} - 1 - izh(x, s))\mu(dx)ds\right), \quad z \in \mathbb{R},$$

where $\Phi = \mathbf{1}$ denotes the vacuum vector in $\Gamma(\mathcal{H})$. In other terms, the spectral measure associated to $\Phi$ of $Y^h$ is the law of $I_1(h) = \int_{M \times \mathbf{R}_+} h(x, t)\tilde{X}(dx, dt)$.

# 3 Operators on Fock space defined by derivation of kernels

Let $\mathbf{1}$ denotes the unit function in $\mathcal{K}$. Let $\partial$ denote the operator of differentiation with respect to $t \in \mathbb{R}^+$ of differentiable functions $f \in L^2(\mathbb{R}_+, \mathcal{K})$ and let $\partial^*$ be defined on $L^2(\mathbb{R}_+, \mathcal{K})$ as

$$\partial^* u(t) = \int_0^t u(s)ds \in \mathcal{K}, \quad t \in \mathbb{R}_+, \ u \in L^2(\mathbb{R}_+, \mathcal{K}).$$

The operators $\partial$, $\partial^*$ are adjoint in the following sense:

$$(\partial u, v)_{L^2(\mathbf{R}_+, \mathcal{K})} = (u, \partial^* v)_{L^2(\mathbf{R}_+, \mathcal{K})}, \quad u \in \mathcal{C}, \ v \in L^2(\mathbb{R}_+, \mathcal{K}).$$

Let $\xi_{[t}$ denote the projection in $L^2(\mathbb{R}_+, \mathcal{K})$ defined as $\xi_{[t} f = 1_{[t,\infty[} f$, $t \in \mathbb{R}_+$.

## 3.1 Operators $\nabla^{\ominus}$ and $\nabla^{\oplus}$

As a convention, tensor products are completed only if vector spaces are closed.

**Definition 2** *We define respectively on $\mathcal{S}$ and $\mathcal{S} \otimes L^2(\mathbb{R}_+)$ the following unbounded operators by linearity and polarization.*

- *Let $\nabla^{\ominus} : \Gamma(L^2(\mathbb{R}_+, \mathcal{K})) \longrightarrow \Gamma(L^2(\mathbb{R}_+, \mathcal{K})) \otimes L^2(\mathbb{R}_+)$ be defined by*

$$\nabla_t^{\ominus} f^{\circ n} = -n f^{\circ(n-1)} \circ \left( \xi_{[t} \partial f \right) \in L^2(\mathbb{R}_+, \mathcal{K})^{\circ n}, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}. \quad (6)$$

- *Let $\nabla^{\oplus} : \Gamma(L^2(\mathbb{R}_+, \mathcal{K})) \otimes L^2(\mathbb{R}_+, \mathcal{K}) \longrightarrow \Gamma(L^2(\mathbb{R}_+, \mathcal{K}))$ be defined by*

$$\nabla^{\oplus} \left( f^{\circ n} \otimes g \right) = n f^{\circ(n-1)} \circ \left( \partial \left( f \partial^* g \right) \right), \quad n \in \mathbb{N}. \quad (7)$$

An operator similar to $\nabla^{\ominus}$ has been defined in different contexts in [11], [21]. Mimicking the quantum probabilistic definition (2) of the operators $a_g^-$ and $a_g^+$, let $a_g^{\ominus}$, $a_g^{\oplus}$, $g \in L^2(\mathbb{R}_+)$ be defined as

$$a_g^{\ominus} F = (\nabla^{\ominus} F, g)_{L^2(\mathbf{R}_+)}, \quad a_g^{\oplus} F = \nabla^{\oplus}(F \otimes g), \quad F \in \mathcal{S}. \quad (8)$$

The definitions of $a_g^{\ominus}$ and $a_g^{\oplus}$ as operators on $\mathcal{S}$ can also be extended to $g \in L^2(\mathbb{R}^+, \mathcal{K})$:

**Definition 3** *For $g \in L^2(\mathbb{R}^+, \mathcal{K})$ we define on $\mathcal{S}$*

$$a_g^{\ominus} F = -\nabla^+(\partial^* g \partial \circ \nabla^- F), \quad a_g^{\oplus} F = \nabla^+(\partial(\partial^* g \nabla^- F)) = \nabla^{\oplus}(F \otimes g). \quad (9)$$

7

This definition is consistent with (8) since for $g \in L^2(\mathbb{R}_+)$, $a_g^\oplus F = \nabla^\oplus(F \otimes g)$, and $a_g^\ominus f^{\circ n} = (\nabla^\ominus f^{\circ n}, g)_{L^2(\mathbf{R}_+)}$, as follows from the equality

$$a_g^\ominus f^{\circ n} = -n f^{\circ(n-1)} \circ (\partial f \partial^* g) = -n \int_0^\infty f^{\circ(n-1)} \circ \left[\xi_{[t}\partial f\right] g(t)dt = (\nabla^\ominus f^{\circ n}, g)_{L^2(\mathbf{R}_+)}, \tag{10}$$

and $a_g^\ominus$ is adjoint of $a_g^\oplus$:

$$
\begin{aligned}
< a_g^\ominus h^{\circ n}, f^{\circ n} >_{\Gamma(\mathcal{H})} &= n < h^{\circ(n-1)} \circ \partial(h \partial^* g), f^{\circ n} >_{\Gamma(\mathcal{H})} \\
&= n(h,f)_{L^2(\mathbf{R}_+, \mathcal{K})}(\partial(h\partial^* g), f)_{L^2(\mathbf{R}_+, \mathcal{K})} \\
&= -n(h,f)_{L^2(\mathbf{R}_+, \mathcal{K})}(h\partial^* g, \partial f)_{L^2(\mathbf{R}_+, \mathcal{K})} = < h^{\circ n}, a_g^\ominus f^{\circ n} >_{\Gamma(\mathcal{H})},
\end{aligned}
$$

$f, g, h \in \mathcal{C}$. Consequently, $\nabla^\ominus$ and $\nabla^\oplus$ are also adjoint of each other:

$$< \nabla^\oplus(f^{\circ n} \otimes g), h^{\circ(n+1)} >_{\Gamma(\mathcal{H})} = < f^{\circ n} \otimes g, \nabla^\ominus h^{\circ n} >_{\Gamma(\mathcal{H}) \otimes L^2(\mathbf{R}_+)},$$

$f, g, h \in \mathcal{C}$. and since $\mathcal{S}$ is dense in $L^2(\mathbb{R}_+, \mathcal{K})$, and $\nabla^\ominus$, $\nabla^\oplus$, $a_g^\ominus$, $a_g^\oplus$ are closable.

## 3.2  Relationship to the number operator

We notice here that the operators $\nabla^\ominus$, $\nabla^\oplus$, and the type of time perturbation they relate to are closely connected to the number operator on $\Gamma(L^2(\mathbb{R}_+, \mathcal{K}))$, or more precisely to a second quantization operator. This property will be useful in Sect. 7 to distinguish between the two different notions of Skorohod integrals.

**Proposition 2** *For $g \in L^2(\mathbb{R}_+, \mathcal{K})$, we have*

$$a_g^\ominus + a_g^\oplus = d\Gamma(g), \tag{11}$$

*on $\mathcal{S}$, where $g$ is identified to a multiplication operator in $L^2(\mathbb{R}_+, \mathcal{K})$.*

*Proof.* This relation is a consequence of the identity $\partial(f\partial^* g) - \partial^* g \partial f = fg$, for $f \in \mathcal{C}$, $g \in L^2(\mathbb{R}_+, \mathcal{K})$, and of the definition (9) of $a_g^\ominus$ and $a_g^\oplus$. $\qquad\square$

This decomposition can be viewed as a decomposition of Poisson noise into creation and annihilation parts, in a way that parallels the well-known decomposition of white noise.

## 3.3 Product rule

In this subsection elements of $\Gamma(\mathcal{H})$ are identified with random variables in $L^2(\Omega)$ via the Wiener-Poisson-Itô isometric isomorphism, hence $(\nabla_{x,t}^- F)_{(x,t) \in M \times \mathbf{R}_+}$ is an element of $L^2(M \times \mathbf{R}_+, L^2(\Omega))$, for $F \in Dom(\nabla^-)$.

**Remark 1** *Due to the Itô formula (5), every element of $\mathcal{S}$ can be expressed as a polynomial in single stochastic integrals with respect to $X$. Conversely, a polynomial in single stochastic integrals with respect to $X$ is in $\mathcal{S}$, provided its integrands are in $\mathcal{C}$.*

As will be recalled in the next section, $\nabla_{0,t}^-$ is identified to a derivation operator, and $\nabla_{x,t}^-$ is, for $x \neq 0$, is identified to a finite difference operator. Hence we have the identity

$$\nabla_{x,t}^-(FG) = F\nabla_{x,t}^- G + G\nabla_{x,t}^- F + 1_{\{x \neq 0\}}\nabla_{x,t}^- F\nabla_{x,t}^- G, \quad (x,t) \in M^* \times \mathbf{R}_+, \qquad (12)$$

$F, G \in \mathcal{S}$.

**Proposition 3** *The action of the operator $a_g^\ominus$ on a product is the following:*

$$a_g^\ominus(FG) = Fa_g^\ominus G + Ga_g^\ominus F - \int_{M \times \mathbf{R}_+} g(x,t)\nabla_{x,t}^- F\nabla_{x,t}^- G\mu(dx)dt, \qquad (13)$$

$g \in L^2(M \times \mathbf{R}_+)$, $F, G \in \mathcal{S}$.

*Proof.* The proof of this result uses the Itô formula, or equivalently Prop. 1. Let $h, f \in \mathcal{C}$, and let $\hat{h}$ denote the function $\hat{h} = \pi h$, where $\pi : L^2(M \times \mathbf{R}_+) \to L^2(M^* \times \mathbf{R}_+)$ is the canonical projection. We have

$$a_g^\ominus\left(I_1(h)I_n(f^{\circ n})\right) + \int_{M \times \mathbf{R}_+} g(x,t)\nabla_{x,t}^- I_1(h)\nabla_{x,t}^- I_n(f^{\circ n})\mu(dx)dt$$

$$= -nI_n\left(f^{\circ(n-1)} \circ \left[(\partial f\hat{h} + f\partial\hat{h})\partial^* g\right]\right)$$

$$-n(n-1)I_n\left(f^{\circ(n-2)} \circ \hat{h}f \circ \partial f\partial^* g\right) - nI_{n+1}\left(f^{\circ(n-1)} \circ h \circ \partial f\partial^* g\right)$$

$$-I_{n+1}\left(f^{\circ n} \circ \partial f\partial^* g\right) - n(n-1)(f,h)_{\mathcal{H}}I_{n-1}\left(f^{\circ(n-2)} \circ \partial f\partial^* g\right)$$

$$+n\int_{M \times \mathbf{R}_+} g(x,t)h(x,t)f(x,t)\mu(dx)dtI_{n-1}(f^{\circ(n-1)})$$

$$= -n(n-1)I_n\left(f^{\circ(n-1)} \circ f\hat{h} \circ \partial f\partial^* g\right) - nI_n\left(f^{\circ(n-1)} \circ \hat{h}\partial f\partial^* g\right)$$

$$-nI_{n+1}\left(f^{\circ(n-1)} \circ h \circ \partial f\partial^* g\right) - n(n-1)(f,h)_{\mathcal{H}}I_{n-1}\left(f^{\circ(n-2)} \circ \partial f\partial^* g\right)$$

$$-n(\partial^* g, h\partial f)_{\mathcal{H}}I_{n-1}\left(f^{\circ(n-1)}\right)$$

$$-nI_n\left(f^{\circ(n-1)} \circ \partial\hat{h}\partial^* gf\right) - I_{n+1}\left(f^{\circ n} \circ \partial h\partial^* g\right) - n(\partial^* g, f\partial h)_{\mathcal{H}}I_{n-1}(f^{\circ(n-1)})$$

$$= I_1(h)a_g^\ominus I_n(f^{\circ n}) + I_n(f^{\circ n})a_g^\ominus I_1(h).$$

The formula is then easily extended by induction to $F, G \in \mathcal{S}$ from Prop. 1, as in Prop. 8 of [22]. Assume that (13) holds for $F = I_n(f^{\circ n})$ and $G = I_1(h)^k$ for some $k \geq 1$. Then using the identity (12) satisfied by $\nabla^-_{x,t}$, we can write

$$
\begin{aligned}
& a^\ominus_g \left( I_n(f^{\circ n}) I_1(h)^{k+1} \right) \\
&= \; I_1(h) a^\ominus_g (I_n(f^{\circ n}) I_1(h)^k) + I_n(f^{\circ n}) I_1(h)^k a^\ominus_g I_1(h) \\
& \quad - \int_{M \times \mathbf{R}_+} g(x,t) \nabla^-_{x,t} I_1(h) \nabla^-_{x,t} (I_1(h)^k I_n(f^{\circ n})) \mu(dx) dt \\
&= \; I_1(h) \left( I_1(h)^k a^\ominus_g I_n(f^{\circ n}) + I_n(f^{\circ n}) a^\ominus_g \left( I_1(h)^k \right) \right) \\
& \quad - I_1(h) \int_{M \times \mathbf{R}_+} g(x,t) \nabla^-_{x,t} \left( I_1(h)^k \right) \nabla^-_{x,t} I_n(f^{\circ n}) \mu(dx) dt + I_n(f^{\circ n}) I_1(h)^k a^\ominus_g I_1(h) \\
& \quad - \int_{M \times \mathbf{R}_+} g(x,t) \nabla^-_{x,t} I_1(h) \left( I_1(h)^k \nabla^-_{x,t} I_n(f^{\circ n}) + I_n(f^{\circ n}) \nabla^-_{x,t} \left( I_1(h)^k \right) \right) \mu(dx) dt \\
& \quad - \int_{M \times \mathbf{R}_+} 1_{\{x \neq 0\}} g(x,t) \nabla^-_{x,t} I_1(h) \nabla^-_{x,t} I_1(h)^k \nabla^-_{x,t} I_n(f^{\circ n}) \mu(dx) dt \\
&= \; I_1(h)^{k+1} a^\ominus_g I_n(f^{\circ n}) + I_n(f^{\circ n}) a^\ominus_g \left( I_1(h)^{k+1} \right) \\
& \quad - \int_{M \times \mathbf{R}_+} g(x,t) \nabla^-_{x,t} \left( I_1(h)^{k+1} \right) \nabla^-_{x,t} I_n(f^{\circ n}) \mu(dx) dt.
\end{aligned}
$$

$\square$

Following the proof of Prop. 3 we can show

$$
\nabla^\ominus_t (FG) = F \nabla^\ominus_t G + G \nabla^\ominus_t F - \int_M \nabla^-_{x,t} F \nabla^-_{x,t} G \mu(dx), \quad F, G \in \mathcal{S}, \ t \in \mathbb{R}_+, \quad (14)
$$

hence for $g \in L^2(\Omega) \otimes L^2(\mathbb{R}_+)$,

$$
a^\ominus_g (FG) = F a^\ominus_g G + G a^\ominus_g F - \int_0^\infty g(t) \int_M \nabla^-_{x,t} F \nabla^-_{x,t} G \mu(dx) dt, \quad (15)
$$

a.s., $F, G \in \mathcal{S}$. However this formula is not extended to a random $g \in L^2(\Omega) \otimes L^2(M \times \mathbb{R}_+)$ since $a^\ominus_g$ is not defined for such processes. See Sect. 7 for an extension of the definition to random $g$.

# 4 Perturbations of Lévy processes and their Fock space interpretation

In this section we study the probabilistic interpretations of $\nabla^-$ and $\nabla^\ominus$. While it is well known that $\nabla^-$ is interpreted by shifts of trajectories on both the Wiener and Poisson spaces, we show that $\nabla^\ominus$ corresponds to perturbations by time changes.

## 4.1 Perturbations by shifts of trajectories

First by perturbation via addition of a jump to the Poisson point measure and infinitesimal shift of the Brownian trajectory, we get the annihilation operator on $\Gamma(\mathcal{H})$.

**Proposition 4** *We have for $F \in \mathcal{S}$:*

$$\nabla_{x,t}^- F = F\left(X(\cdot) + \delta_{x,t}(\cdot)\right) - F, \quad (x,t) \in M^* \times \mathbb{R}_+, \tag{16}$$

*and*

$$\nabla_{0,t}^- f\left(I_1(h_1), \ldots, I_1(h_n)\right) = \sum_{i=1}^{i=n} h_i(0,t)\partial_i f\left(I_1(h_1), \ldots, I_1(h_n)\right), \tag{17}$$

$f \in \mathcal{C}_b^1(\mathbb{R}^n)$, $h_1, \ldots, h_n \in \mathcal{C}$, $t \in \mathbb{R}_+$, *or more formally:*

$$\nabla_{0,t}^- F = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(F\left(X(\cdot) + \varepsilon\delta_{0,t}(\cdot)\right) - F\right), \quad F \in \mathcal{S}.$$

*Proof.* The Wiener and Poisson parts of $\tilde{X}$ can be treated separately. In the Wiener case we refer to [14], [27] and to the references therein. In the Poisson case this result is contained in [8], [16]. $\qquad \square$

The Wiener part $(\nabla_{0,t} F)_{t \in \mathbb{R}_+}$ of the operator $\nabla^-$ is also called the Malliavin derivative, cf. [14]. The following relation between the values of the gradients $\nabla^-$ on $M^* \times \mathbb{R}_+$ and on $\{0\} \times \mathbb{R}_+$ is connected to the convergence of the renormalized Poisson process to Brownian motion, and shows that $\nabla^- F$ on $\{0\} \times \mathbb{R}_+$ can be obtained by continuity from its values on $M^* \times \mathbb{R}_+$.

**Proposition 5** *Let $h \in \mathcal{C}_c(M)$ with $\lim_{x \to 0} h(x) = 0$ and $h(x) \neq 0$, $\forall x \in M^*$. If $F \in Dom(\nabla^-)$ is of the form $F = f\left(\int_M \int_0^t h(x)\tilde{X}(dx, ds)\right)$, $f \in \mathcal{C}_b^1(\mathbb{R})$, $h \in \mathcal{C}_c(M)$, then*

$$\nabla_{0,s}^- F = \lim_{x \to 0} \frac{1}{h(x)} \nabla_{x,s}^- F, \quad s \in \mathbb{R}_+, \quad a.s.$$

*Proof.* This is a consequence of Relations (16) and (17). $\qquad \square$

## 4.2 Perturbations by time changes

We now turn to the probabilistic interpretation of $\nabla^\ominus$ and $\nabla^\oplus$. The main difference between our construction and that of e.g. [1] is that only jump times are perturbed in the Poisson part of $X$. In this way we retain the connection between variational calculus and stochastic integration, cf. Sect. 7. Since this probabilistic interpretation will involve time changes and such perturbations can not be absolutely continuous with respect to the Wiener measure (they can be, however, with respect to the Poisson measure), we will consider functionals that can be defined everywhere, i.e. for every trajectory of $X$. Single stochastic integrals with respect to a Poisson measure can be defined everywhere provided the support of the integrand has finite intensity measure. Wiener single stochastic integrals can be defined for every Brownian trajectory if the integrand is continuously differentiable in the time variable so as to allow to write an integration by parts formula. Hence single stochastic integrals in $\mathcal{S}$ can be defined trajectory by trajectory, but $\mathcal{S}$ also contains iterated stochastic integrals whose definition trajectory by trajectory is a priori ambiguous. We choose to define them everywhere by taking into account Remark 1 and by using the expression of elements of $\mathcal{S}$ as polynomials in single stochastic integrals. More precisely, we state the following definitions.

**Definition 4** *Let $F$ be a random variable defined for every trajectory of $X$. For $h \in L^2(M \times \mathbb{R}_+) \cap L^\infty(M \times \mathbb{R}_+)$ with $\| h \|_{L^\infty(M \times \mathbf{R}_+)} < 1$, let $\mathcal{T}_h F$ denote the functional $F \in \mathcal{S}$ evaluated at time-changed trajectories whose jumps are obtained from the jumps of $N(dx, ds)$ via the mapping*

$$
\begin{aligned}
M \times \mathbb{R}_+ & \longrightarrow M \times \mathbb{R}_+ \\
(x, t) & \mapsto (x, \nu_h(x, t)) = (x, t + \partial^* h(x, t)),
\end{aligned}
$$

*and whose continuous part is given by the time-changed Brownian motion $(B^h_t)_{t \in \mathbf{R}_+}$ defined as*

$$
B^h_{\nu_h(0,t)} = B_t, \quad t \in \mathbb{R}_+.
$$

Since most functionals of stochastic analysis are only defined almost surely, we will also need the following.

**Definition 5** *Let $\mathcal{D}$ denote the vector space dense in $L^2(\Omega)$ generated by*

$$
\{ I_n(h_1 \circ \cdots \circ h_n) \; : \; h_1, \dots, h_n \in \cap_{p \geq 2} L^p(M \times \mathbb{R}_+), \; n \in \mathbb{N} \}.
$$

Let $h \in L^2(M \times \mathbb{R}_+) \cap L^\infty(M \times \mathbb{R}_+)$ *with* $\| h \|_{L^\infty(M \times \mathbb{R}_+)} < 1$. *For* $F \in \mathcal{D}$ *of the form* $I_n(f_1 \circ \cdots \circ f_n)$, *let* $F = f(I_1(g_1), \ldots, I_1(g_m))$ *denote the expression of $F$ as a polynomial in single stochastic integrals obtained from Prop. 1. We define*

$$\mathcal{U}_h F = f(I_1(g_1 \circ \nu_h), \ldots, I_1(g_m \circ \nu_h)).$$

*The definition of $\mathcal{U}_h$ extends to $\mathcal{D}$ by linearity.*

The interest in the operator $\mathcal{U}_h$, compared to $\mathcal{T}_h$, is that it is defined on a set of $L^2$ functionals, whereas $\mathcal{T}_h$ is not. The link between $\mathcal{U}_h$ and $\mathcal{T}_h$ is given by the following remark.

**Remark 2** *For any $F \in \mathcal{S}$ there is a version $\hat{F}$ of $F$ such that $\mathcal{U}_h F = \mathcal{T}_h \hat{F}$, a.s.*

*Proof.* It suffices to do the proof in the Wiener case, for $F \in \mathcal{S}$ of the form $F = I_1(f)$. It this case, $I_1(f) = \int_0^\infty f'(s) B_s ds$, a.s., hence letting $\hat{F} = \int_0^\infty f'(s) B_s ds$, we obtain

$$\mathcal{T}_h \hat{F} = \int_0^\infty f'(s) B(\nu_h^{-1}(s)) ds = \int_0^\infty f'(\nu_h(s)) B_s (1 + h(s)) ds = \int_0^\infty f(\nu_h(s)) dB_s, \quad a.s.,$$

hence $\mathcal{T}_h \hat{F} = \mathcal{U}_h F$, a.s.

$\square$

**Proposition 6** *Let $u \in L^2(M \times \mathbb{R}_+) \cap L^\infty(M \times \mathbb{R}_+)$. For $F \in \mathcal{S}$ we have*

$$-\frac{d}{d\varepsilon} \mathcal{U}_{\varepsilon u} F_{|\varepsilon=0} = a_u^\ominus F + \int_{M \times \mathbb{R}_+} u(x,t) \left( 1_{\{x \neq 0\}} \nabla_{x,t}^- + \frac{1}{2} 1_{\{x=0\}} \nabla_{0,t}^- \nabla_{0,t}^- \right) F \mu(dx) dt, \tag{18}$$

*the limit being taken in $L^2(\Omega)$.*

*Proof.* Relation (18) is proved in two steps. First we notice that it holds for a simple stochastic integral $I_1(h)$, $h \in \mathcal{C}$, cf. Prop. 9 of [22] and [21], and then use the product rules (13) and (15) which imply that

$$F \mapsto a_u^\ominus F + \int_{M \times \mathbb{R}_+} u(x,t) \left( 1_{\{x \neq 0\}} \nabla_{x,t}^- + \frac{1}{2} 1_{\{x=0\}} \nabla_{0,t}^- \nabla_{0,t}^- \right) F \mu(dx) dt$$

is a derivation operator on $\mathcal{S}$, given that $\nabla_{0,t}^-$ is a derivation operator and that $\nabla_{x,t}^-$ for $x \neq 0$ satisfies as a finite difference operator:

$$\nabla_{x,t}^-(FG) = F \nabla_{x,t}^- G + G \nabla_{x,t}^- F + \nabla_{x,t}^- F \nabla_{x,t}^- G, \quad (x,t) \in M^* \times \mathbb{R}_+.$$

□

Prop. 6 will be interpreted in Sect. 5 as an extended form of the Itô formula, in which $a_g^{\ominus}F$, having expectation zero, represents a "martingale term". Although it is dense in $L^2(\Omega)$, the class $\mathcal{S}$ is too small to be of real interest in stochastic analysis since it does not contain the increments of $X$. Thus we need to extend Prop. 6 to a wider class of functionals.

**Definition 6** *We define the operator* $\mathcal{A} : \mathcal{D} \to L^2(\Omega) \otimes L^2(\mathbb{R}_+)$ *by*

$$\mathcal{A}_s F = \int_M \left( 1_{\{x \neq 0\}} \nabla_{x,s}^- F + \frac{1}{2} 1_{\{x=0\}} \nabla_{0,s}^- \nabla_{0,s}^- F \right) \mu(dx), \quad dP \otimes ds \ a.e.$$

The operator $\mathcal{A}$, (whose "Wiener part" is the Gross Laplacian after integration with respect to $ds$), will be used to define a notion of pseudo generator for non-Markovian processes that will, due to Relation (24) below, extend the classical notion of generator. The operator $\mathcal{A}$ is "intrinsic", in that unlike classical generators, it is not determined by a particular process. Absolutely continuous drifts are not considered here because their influence is of a deterministic nature and for this reason they do not create new problems in an extension of stochastic calculus to an anticipative or to a non-Markovian setting. Similarly, in order to simplify the exposition, stochastic integrals with respect to the Poisson measure are evaluated for functions with finite intensity measure support. The general case can be treated by introduction of appropriate compensators.

**Proposition 7** *We have for* $u \in \mathcal{C}_c^1(\mathbb{R}_+)$ *and* $F \in \mathcal{D}$:

$$-\frac{d}{d\varepsilon} < \mathcal{U}_{\varepsilon u} F, G >_{|\varepsilon=0} = < F, a_u^{\oplus} G >_{L^2(\Omega)} + < \mathcal{A}F, G \otimes u >_{L^2(\Omega) \otimes L^2(\mathbb{R}_+)}, \quad G \in \mathcal{S}. \quad (19)$$

*Proof.* By comparison with the Malliavin calculus by space perturbation of trajectories, the difficulty lies here in the fact that on Wiener space the transformation $\mathcal{T}_{\varepsilon u}$ is not absolutely continuous. By polarization and use of the Itô formula (5) it is sufficient to prove (19) for $F = I_1(f)^n$, where $f \in \cap_{p \geq 2} L^p(M \times \mathbb{R}_+)$. We have the chaos expansion

$$I_1(f)^n = \sum_{j=0}^{j=n} I_j(f_j).$$

Due to the multiplication formula for multiple Wiener-Poisson stochastic integrals, $f_k$ is of the form $f_k = h_1^k \circ \cdots \circ h_k^k \times P_k(f)$, where $h_1^k, \ldots, h_k^k$ are powers of $f$ of degree

lower than $2n$, and $P_k(f)$ is a polynomial in integrals on $M \times \mathbb{R}_+$ of powers of $f$. For $h \in L^p(M \times \mathbb{R}_+)$, $p \geq 2$, we let $h(\nu_{\varepsilon u})$ be the function defined by $(x, t) \mapsto h(x, \nu_{\varepsilon u}(t))$. For $\varepsilon$ in a certain neighborhood of zero, $(x, t) \mapsto (x, \nu_{\varepsilon u}(t))$ is invertible and absolutely continuous with bounded Radon-Nykodim derivative, hence $h(\nu_{\varepsilon u})$ is well-defined in $L^p(M \times \mathbb{R}_+)$. We have

$$\mathcal{U}_{\varepsilon h} F = \sum_{j=0}^{j=n} P_j(f(\nu_{\varepsilon u})) I_j(h_1^j(\nu_{\varepsilon u}) \circ \cdots \circ h_n^j(\nu_{\varepsilon u})).$$

We assume that $G$ is in the $k$-th chaos, $k \leq n$, and that it is written as $G = I_k(g^{\circ k})$, $g \in \mathcal{C}$. With this notation,

$$< \mathcal{U}_{\varepsilon u} F, G >_{L^2(\Omega)} = P_k(f(\nu_{\varepsilon u}))(h_1^k(\nu_{\varepsilon u}) \circ \cdots \circ h_k^k(\nu_{\varepsilon u}), g^{\circ k})_{L^2(M \times \mathbf{R}_+)^{\otimes k}}. \qquad (20)$$

Now,

$$(h_1^k(\nu_{\varepsilon u}) \circ \cdots \circ h_k^k(\nu_{\varepsilon u}), g^{\circ k})_{L^2(M \times \mathbf{R}_+)^{\otimes k}}$$
$$= \left( h_1^k, \left( \frac{g}{1 + \varepsilon u} \right) (\nu_{\varepsilon u}^{-1}) \right)_{L^2(M \times \mathbf{R}_+)} \cdots \left( h_k^k, \left( \frac{g}{1 + \varepsilon u} \right) (\nu_{\varepsilon u}^{-1}) \right)_{L^2(M \times \mathbf{R}_+)}, \quad (21)$$

and the derivative of $\left( \frac{g}{1+\varepsilon u} \right) (\nu_{\varepsilon u}^{-1})$ in $\varepsilon$ is continuously differentiable and uniformly bounded on $M \times \mathbb{R}_+$ for $\varepsilon$ in a neighborhood of zero by a function integrable on $M \times \mathbb{R}_+$. An analogous change of variables can be performed in $P_k(f)$, hence $< \mathcal{U}_{\varepsilon u} F, G >$ is differentiable in $\varepsilon$ in a certain neighborhood of zero. For $F \in \mathcal{S}$, Relation (19) is a consequence of Prop. 6 and of the duality between $\nabla^{\ominus}$ and $\nabla^{\oplus}$. In order to prove (19) for $F \in \mathcal{D}$ we need to exchange the derivation with respect to $\varepsilon$ with the limit of a sequence $(F_n)_{n \in \mathbf{N}}$ in $\mathcal{S}$ such that $(F_n)_{n \in \mathbf{N}}$ and $(\mathcal{A} F_n)_{n \in \mathbf{N}}$ converge respectively in $L^2(\Omega)$ and $L^2(\Omega) \otimes L^2(\mathbb{R}_+)$ to $F \in \mathcal{D}$ and to $\mathcal{A} F$. Hence the proposition will hold if we show

$$\frac{d}{d\varepsilon} < \mathcal{U}_{\varepsilon u} F, G >_{|\varepsilon = 0} = \lim_{n \to \infty} \frac{d}{d\varepsilon} < \mathcal{U}_{\varepsilon u} F_n, G >_{|\varepsilon = 0}, \quad G \in \mathcal{S}.$$

For this it is sufficient to prove the following, given the polynomials expressions (20), (21) . Let $N \geq 1$ and let $(\alpha_n)_{n \in \mathbf{N}}$ be a sequence in $\mathcal{C}$ converging in $\cap_{p=2}^{p=N} L^p(M \times \mathbb{R})$ to $\alpha$. We have for $\varepsilon$ in a neighborhood of zero

$$\frac{d}{d\varepsilon}(\alpha(\nu_\varepsilon), g)_{L^2(M \times \mathbf{R}_+)}$$
$$= \frac{d}{d\varepsilon}(\alpha, \left( \frac{g}{1 + \varepsilon u} \right) (\nu_{\varepsilon u}^{-1}))_{L^2(M \times \mathbf{R}_+)} = (\alpha, \frac{d}{d\varepsilon} \left( \frac{g}{1 + \varepsilon u} \right) (\nu_{\varepsilon u}^{-1}))_{L^2(M \times \mathbf{R}_+)}$$

15

$$= \lim_{n\to\infty} (\alpha_n, \frac{d}{d\varepsilon}\left(\frac{g}{1+\varepsilon u}\right)(\nu_{\varepsilon u}^{-1}))_{L^2(M\times\mathbf{R}_+)} = \lim_{n\to\infty} \frac{d}{d\varepsilon}(\alpha_n, \left(\frac{g}{1+\varepsilon u}\right)(\nu_{\varepsilon u}^{-1}))_{L^2(M\times\mathbf{R}_+)}$$

$$= \lim_{n\to\infty} \frac{d}{d\varepsilon}(\alpha_n(\nu_{\varepsilon u}), g)_{L^2(M\times\mathbf{R}_+)}.$$

$\square$

Prop. 7 also gives:

$$-\frac{d}{d\varepsilon} < \mathcal{U}_{\varepsilon u}f(I_1(h)), G >_{|\varepsilon=0} \quad = \quad < f(I_1(h)), \nabla^{\oplus}(G\otimes u) >_{L^2(\Omega)}$$

$$+ < \int_0^\infty u_s \left[\mathcal{G}_s^h f\right](I_1(h))ds, G >_{L^2(\Omega)},$$

$G \in \mathcal{S}$, for $f$ polynomial, $h \in \cap_{p\geq 2}L^p(M\times\mathbb{R}_+)$, $u \in \mathcal{C}_c^1(\mathbb{R}_+)$. For $G = 1$ we can along the lines of the proof of Prop. 7 show the following result.

**Proposition 8** *Let $T \in \mathbb{R}_+$ and let $(u_\varepsilon)_{\varepsilon\in\mathbb{R}_+} \subset \mathcal{C}_c^1([0,T])$ be continuous in $\varepsilon$ for the $\|\cdot\|_{L^\infty(\mathbb{R}_+)}$ norm. We have for $F \in \mathcal{D}$:*

$$\frac{d}{d\varepsilon}E[\mathcal{U}_{\varepsilon u_\varepsilon}F]_{|\varepsilon=0} = E[(\mathcal{A}F, u_0)_{L^2(\mathbb{R}_+)}].$$

# 5    A chaos approach to the Itô formula

The aim of this section is to develop from Prop. 6 a formula that extends the Itô formula and gives the chaos expansion of its martingale term. The generator $(\mathcal{G}_s^h)_{s\in\mathbb{R}_+}$ of the uncompensated process

$$X_t^h = \int_{M^*} \int_0^t h(x,s)N(dx,ds) + \int_0^t h(0,s)dB_s, \quad t \in \mathbb{R}_+,$$

where $h \in L^2(M\times\mathbb{R}_+)$ has finite measure support, is given by

$$\left[\mathcal{G}_s^h f\right](x) = \int_{M^*} (f(x+h(y,s)) - f(x))\,\mu(dy) + \frac{1}{2}h(0,s)^2\partial^2 f(x), \quad x,s \in \mathbb{R}_+, \quad (22)$$

$f \in \mathcal{C}^2(\mathbb{R})$. The Dynkin formula says that the process $f(X_t^h) - f(0) - \int_0^t \mathcal{G}_s^h f(X_s^h)ds$ is a martingale relative to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, and the Itô formula identifies this martingale:

$$f(X_t^h) - f(0) - \int_0^t \left[\mathcal{G}_s^h f\right](X_s^h)ds$$

$$= \int_0^t h(0,s)f'(X_s^h)dB_s + \int_{M^*}\int_0^t (f(X_s^h + h(x,s)) - f(X_s^h))\tilde{N}(dx,ds), \quad (23)$$

$f \in \mathcal{C}^2(\mathbb{R})$. Note that from (16), (17) and (22) we have the relations

$$\left[ \mathcal{G}_s^h f \right] (I_1(h)) = \int_M \left[ 1_{\{x \neq 0\}} \nabla_{x,s}^- + \frac{1}{2} 1_{\{x=0\}} \nabla_{0,s}^- \nabla_{0,s}^- \right] f(I_1(h)) \mu(dx), \qquad (24)$$

and

$$1_{[0,t]}(s) \left( \mathcal{G}_s^h f \right) (X_t^h) = \int_M \left[ 1_{\{x \neq 0\}} \nabla_{x,s}^- + \frac{1}{2} 1_{\{x=0\}} \nabla_{0,s}^- \nabla_{0,s}^- \right] f(X_t^h) \mu(dx), \qquad (25)$$

for $f \in \mathcal{C}_b^2(\mathbb{R})$. A similar relation can be written if $X^h$ is replaced by a diffusion process but it is not as straightforward, cf. Relation (29) in Sect. 6. The annihilation operator $\nabla^-$ does not appear only in the generator $\mathcal{G}_s^h$, but also in the martingale term of the Itô formula from its expressions (16) and (17), moreover the anticipating Itô formula makes use of $\nabla^-$ in the Wiener case, cf. [14], [26]. However, the martingale term (23) can not be explicitly written with $\nabla^-$. The closest result that directly uses $\nabla^-$ may be the Clark formula, cf. [7], [25] and Sect. 8 for its extension to Lévy processes. The aim of the following lemma is to provide a chaos form for the Itô formula, using the operators $\nabla^\ominus$ and $\nabla^-$.

**Lemma 1** Let $h \in \mathcal{C}$, and let $u \in L^2(\Omega \times \mathbb{R}_+) \cap L^\infty(\Omega \times \mathbb{R}_+)$ be $(\mathcal{F}_t)$-adapted. We have for $f$ polynomial:

$$-\frac{d}{d\varepsilon} \mathcal{U}_{\varepsilon u} f(I_1(h))_{|\varepsilon=0} = (\nabla^\ominus f(I_1(h)), u)_{L^2(\mathbb{R}_+)} + (\mathcal{G}^h f(I_1(h)), u)_{L^2(\mathbb{R}_+)}. \qquad (26)$$

*Proof.* We apply Prop. 6 and use Relation (24). $\qquad \square$

Each term in the above Lemma belongs to $L^2$, but the smoothness imposed on functionals and the type of perturbation chosen do not make obvious the analogy with the Itô formula. The following proposition gives from Lemma 1 more precise information on the links between Itô differentials and the chaotic calculus induced by $\nabla^\ominus$. The formula applies to $X_t^h$ which does not have the smoothness property required in Lemma 1, without having recourse to generalized random variables.

**Proposition 9** Let $h \in \mathcal{C}$, and $X_t^h = \int_M \int_0^t h(x,s) X(dx, ds)$, $t \in \mathbb{R}_+$. We have for $f$ polynomial

$$\lim_{n \to \infty} \int_0^t \nabla_s^\ominus f(X_s^{h,n}) ds = \int_0^t f'(X_s^h) dB_s + \int_0^t \int_M (f(X_s^h + h(x,s)) - f(X_s^h)) \tilde{N}(dx, ds),$$

*where the limit is taken in $L^2(\Omega)$, and $(X^{h,n})_{n \in \mathbf{N}}$ is any sequence converging to $X^h$ in $L^2(\Omega) \otimes L^2(\mathbb{R}_+)$, of the form*

$$X_t^{h,n} = \int_M \int_0^\infty h(x,s)e_n(s-t)X(dx,ds), \quad n \in \mathbf{N}, \ t \in \mathbb{R}_+,$$

*where $(e_n)_{n \in \mathbf{N}} \subset \mathcal{C}_b^1(\mathbb{R})$ converges pointwise to $1_{]-\infty,0]}$ with $e_n(s) = 1$, $s \leq 0$, $0 \leq e_n(s) \leq 1$, $s \geq 0$, $n \in \mathbf{N}$.*

*Proof.* Let $(e_n)_{n \in \mathbf{N}}$ be a sequence of smooth positive functions bounded by one and converging everywhere to $1_{]-\infty,0]}$, with $e_n' = 0$ on $\mathbb{R}_-$. We have almost surely

$$-\frac{\partial}{\partial s}X_s^{h,n} = \int_M \int_0^\infty h(x,u)e_n'(u-s)X(dx,du) = \int_M \int_s^\infty h(x,u)e_n'(u-s)X(dx,du)$$

$$= \int_M \int_s^\infty \frac{\partial}{\partial u}(h(x,u)e_n(u-s))(u)X(dx,du) - \int_M \int_s^\infty \partial h(x,u)e_n(u-s)X(dx,du),$$

which implies

$$\frac{\partial}{\partial s}f(X_s^{h,n}) = -\int_M \int_s^\infty \frac{\partial}{\partial u}(h(x,u)e_n(u-s))X(dx,du)f'(X_s^{h,n})$$
$$+ \int_M \int_s^\infty \partial h(x,u)e_n(u-s)X(dx,du)f'(X_s^{h,n}).$$

With the same argument as in Prop. 6 we can show, using (14) and (24), that

$$-\int_M \int_s^\infty \frac{\partial}{\partial u}(h(x,u)e_n(u-s))X(dx,du)f'(X_s^{h,n}) = \nabla_s^\ominus f(X_s^{h,n}) + \mathcal{G}_s^h f(X_s^{h,n}),$$

hence

$$\frac{\partial}{\partial s}f(X_s^{h,n}) = \nabla_s^\ominus f(X_s^{h,n}) + \mathcal{G}_s^h f(X_s^{h,n}) + \int_M \int_s^\infty \partial h(x,u)e_n(u-s)X(dx,du)f'(X_s^{h,n}),$$

and by integration on $[0,t]$:

$$f(X_t^{h,n}) = f(0) + \int_0^t \nabla_s^\ominus f(X_s^{h,n}) + \int_0^t \mathcal{G}_s^h f(X_s^{h,n})ds$$
$$+ \int_0^t \int_M \int_s^\infty \partial h(x,u)e_n(u-s)X(dx,du)f'(X_s^{h,n})ds.$$

It remains to take the limit in $L^2(\Omega)$, which does not depend on the choice of the sequence $(e_n)_{n \in \mathbf{N}}$ as $n$ goes to infinity, and to use (23). $\qquad\square$

The result of the above proposition might be formally written as

$$\nabla_t^\ominus f(X_{t+}^h) = f'(X_t^h)\partial_t B(t) + \int_M (f(X_t^h + h(x,t)) - f(X_t^h))\partial_t \tilde{N}(dx,t),$$

where $\partial_t B(t)$, $\partial_t \tilde{N}(dx, t)$ denote the Gaussian and Poissonian white noises.

We will now write a statement which is closer than Lemma 1 to the classical Itô formula, and applies to a class of processes that can be anticipating with respect to the Lévy filtration, or non-Markovian. More generally, the considered processes do not need to possess any particular property with respect to a filtration. We state below the properties that should be satisfied by $(X_t)_{t \in \mathbf{R}_+}$ in order to extend the Itô formula and the notion of generator to this process. Apart from the smoothness and integrability hypothesis (1) and (2), the third condition ensures a form of consistency in the time evolution of the process without requiring it to be Markov or adapted.

**Definition 7** *We denote by $\mathcal{V}$ the class of processes $(X_t, (u_\varepsilon^t)_{\varepsilon \in \mathbf{R}_+})_{t \in \mathbf{R}_+}$ where $(X_t)_{t \in \mathbf{R}_+}$ is a family of random variables and $(u_\varepsilon^t)_{t, \varepsilon \in \mathbf{R}_+}$ is a family of functions such that for any $t > 0$,*

1. *$X_t \in \mathcal{D}$,*

2. *$(u_\varepsilon^t)_{\varepsilon \in \mathbf{R}_+}$ is continuous in $\varepsilon$ for the $\| \cdot \|_{L^\infty(\mathbf{R}_+)}$ norm,*

3. *for some $T^t \in \mathbf{R}_+$, $u_\varepsilon^t \in \mathcal{C}_c^1([0, T^t])$ and satisfies $\mathcal{U}_{\varepsilon u_\varepsilon^t} X_t = X_{t-\varepsilon}$ a.s., for $\varepsilon$ in a neighborhood of zero.*

The family $(u_\varepsilon^t)_{t, \varepsilon \in \mathbf{R}_+}$ may be independent of $\varepsilon$, and in this case we use the notation $(X_t, u^t)_{t \in \mathbf{R}_+} \in \mathcal{V}$.

**Theorem 1** *Let $(X_t, u^t)_{t \in \mathbf{R}_+} \in \mathcal{V}$ be such that $X_t \in \mathcal{S}$, $\forall t \in \mathbb{R}_+$. We have the extension of the Itô formula*

$$f(X_t) = f(0) + \int_0^t (\nabla^\ominus f(X_s), u^s)_{L^2(\mathbf{R}_+)} ds + \int_0^t (\mathcal{A} f(X_s), u^s)_{L^2(\mathbf{R}_+)} ds, \qquad (27)$$

*for $f$ polynomial.*

*Proof.* This relation is (in differential form) a consequence of Def. 7 and Prop. 6. $\square$

In (27) the "martingale" term $\int_0^t (\nabla^\ominus f(X_s), u^s)_{L^2(\mathbf{R}_+)} ds$ is actually a finite variation process since $(X_t)_{t \in \mathbf{R}_+} \subset \mathcal{S}$. In the general case, this process is obviously not a martingale but it has expectation zero and by analogy with classical diffusions, a "martingale property" could be written here as

$$E \left[ \int_0^t (\nabla^\ominus f(X_s), u^s)_{L^2(\mathbf{R}_+)} ds \mid \mathcal{F}_v \right] = E \left[ \int_0^t (1_{[0,v]} \nabla^\ominus f(X_s), u^s)_{L^2(\mathbf{R}_+)} ds \mid \mathcal{F}_v \right],$$

$v \in \mathbf{R}_+$, given the property $E[\nabla_s^\ominus F \mid \mathcal{F}_t] = 0$ of $\nabla^\ominus$, $s < t$, obtained from (6).

# 6 Pseudo generators of non-Markovian stochastic processes

In this section we show that Prop. 8 yields a systematic method to find a partial differential or integro-differential equation associated to the law of non-Markovian stochastic processes. In the Markov case the results coincide with the ones obtained via the classical theory. From the operator $\mathcal{A}$ we construct a pseudo generator $\mathcal{L}$ associated to a stochastic process in the class $\mathcal{V}$ of Def. 7.

**Definition 8** *To any process $(X_t, (u_\varepsilon^t)_{\varepsilon \in \mathbb{R}_+})_{t \in \mathbb{R}_+}$ in $\mathcal{V}$ we associate a family $(\mathcal{L}_t)_{t \in \mathbb{R}_+}$ of operators defined as*

$$(\mathcal{L}_t f)(X_t) = E\left[ (\mathcal{A}f(X_t), u_0^t)_{L^2(\mathbb{R}_+)} \mid X_t \right],$$

*for $f$ polynomial, $t \in \mathbb{R}_+$.*

Since $(\mathcal{A}f(X_t), u_0^t)_{L^2(\mathbb{R}_+)} \in L^1(\Omega)$, $\mathcal{L}_t f$ is defined $dp_t$-a.e., where $p_t$ is the law of $X_t$. The following result uses the operator $\mathcal{L}$ to extend the notion of generator.

**Theorem 2** *Let $(X_t, (u_\varepsilon^t)_{\varepsilon \in \mathbb{R}_+})_{t \in \mathbb{R}_+} \in \mathcal{V}$. The law $p_t$ of $X_t$ satisfies the integro-differential equation*

$$\frac{\partial}{dt} \int_{\mathbb{R}} f \, dp_t = \int_{\mathbb{R}} \mathcal{L}_t f \, dp_t, \quad t \in \mathbb{R}_+, \tag{28}$$

*for $f$ polynomial.*

*Proof.* Relation (28) is a direct consequence of Def. 8 and Prop. 8 that give

$$\frac{d}{dt} E[f(X_t)] = E[(\mathcal{A}f(X_t), u^t)], \quad t \in \mathbb{R}_+. \quad \square.$$

Relation (28) can be written in distribution sense as $\frac{\partial}{\partial t} p_t(x) = \mathcal{L}_t^* p_t(x)$, $t \in \mathbb{R}_+$, $x \in \mathbb{R}$, where $\mathcal{L}_t^*$ denotes the adjoint of $\mathcal{L}_t$. In view of the remark at the end of Sect. 4 it is also possible to choose $u^t \in L^2(M \times \mathbb{R}_+)$. In this way we can, for instance in case $X$ is a sum of independent Poisson processes, perturb each process independently.

Until the end of this section we focus on the continuous case, in which it is possible to gain more information on the operator $\mathcal{L}_t$. We have

$$\mathcal{A}_s f(X_t) \;=\; \frac{1}{2}\nabla_{0,s}^-(f'(X_t)\nabla_{0,s}^- X_t) = \frac{1}{2}f''(X_t)(\nabla_{0,s}^- X_t)^2 + \frac{1}{2}f'(X_t)\nabla_{0,s}^- \nabla_{0,s}^- X_t.$$

Hence $\mathcal{L}_t$ is of the form

$$\mathcal{L}_t = a_t(x)\partial_x^2 + b_t(x)\partial_x,$$

where $a_t$ and $b_t$ are functions defined $p_t$-a.e. on $\mathbb{R}$ as

$$a_t(X_t) = \frac{1}{2}E[((\nabla^-_{0,\cdot}X_t)^2, u^t)_{L^2(\mathbb{R}_+)} \mid X_t], \quad b_t(X_t) = \frac{1}{2}E[(\nabla^-_{0,\cdot}\nabla^-_{0,\cdot}X_t, u^t)_{L^2(\mathbb{R}_+)} \mid X_t].$$

We now illustrate Th. 2 with examples whose choices relies essentially on the simplicity of calculations. In most cases, various differential equations can be written for the law $p_t$ of a stochastic process, even when the Itô formula is not directly applicable. The point in the present method is that it gives a systematic procedure to derive a differential equation which is "canonical" in that its coincides with the result obtained via the Itô formula in the Markov case. For each $a, b \in \mathbb{R}_+$, $a < b$, we choose a function $e_{a,b} \in \mathcal{C}^\infty_c(\mathbb{R}_+)$ with support in $[a, b]$ and such that $\int_a^b e_{a,b}(x)dx = 1$, with $e_{a,a} = 0$.

- The main interest in this example is that it does not require the computation of a conditional expectation, hence it clearly shows the role played by $\nabla^-$. Let $X_t = B_t^2 + (B_{2t} - B_t)^2$, $t \in \mathbb{R}_+$. The process $(X_t)_{t \in \mathbb{R}_+}$ is not Markovian, but it has same law as a squared Bessel process. Let $u^t = e_{0,t/2} + e_{t,3t/2}$, $t > 0$. We have $\mathcal{U}_{\varepsilon u^t}X_t = X_{t-\varepsilon}$, hence $(X_t, u^t)_{t \in \mathbb{R}_+} \in \mathcal{V}$, and

$$
\begin{aligned}
\mathcal{A}_s f(X_t) &= \frac{1}{2}\nabla^-_{0,s}\nabla^-_{0,s}f(X_t) \\
&= \frac{1}{2}\nabla^-_{0,s}\left(21_{[0,t]}(s)B_t f'(X_t) + 21_{[t,2t]}(s)(B_{2t} - B_t)f'(X_t)\right) \\
&= 1_{[0,2t]}(s)(f'(X_t) + 2B_t^2 1_{[0,t]}(s) + 2(B_{2t} - B_t)^2 1_{[t,2t]}(s))f''(X_t),
\end{aligned}
$$

hence $(\mathcal{A}f(X_t), u^t)_{L^2(\mathbb{R}_+)} = 2X_t f''(X_t) + 2f'(X_t)$, and $\mathcal{L}_t = 2x\partial_x^2 + 2\partial_x$. We retrieved in this way the partial differential equation $\partial_t f_t = 2x\partial_x^2 f_t + 2\partial_x f_t$ satisfied by the density $f_t$ of the law $p_t$ of $(X_t)_{t \in \mathbb{R}_+}$.

- This example shows that the coefficient of the second order derivative term of the pseudo generator is allowed to be negative in our approach. Consider the process $X_t = \int_{a(t)}^{b(t)} h(s)dB_s$ where $a, b \in \mathcal{C}^1(\mathbb{R}_+)$, $0 \le a < b$, and $h \in L^\infty(\mathbb{R}_+)$ is a step function $h = \sum_{i=1}^{i=n} \alpha_i 1_{[t_i, t_{i+1}[}$, and define $u^t_\varepsilon$ as

$$
\begin{aligned}
u^t_\varepsilon &= \sum_{i=1}^{i=n} 1_{]t_i, t_{i+1}[}(a(t))\left(\frac{a(t+\varepsilon) - a(t)}{\varepsilon}e_{t_i, a(t)} - \frac{a(t+\varepsilon) - a(t)}{\varepsilon}e_{a(t), t_{i+1}}\right) \\
&\quad + 1_{]t_i, t_{i+1}[}(b(t))\frac{b(t+\varepsilon) - b(t)}{\varepsilon}e_{t_i, b(t)},
\end{aligned}
$$

for $\varepsilon > 0$, and for $\varepsilon = 0$ as:

$$u_0^t = \sum_{i=1}^{i=n} 1_{]t_i,t_{i+1}[}(a(t))\left(a'(t)e_{t_i,a(t)} - a'(t)e_{a(t),t_{i+1}}\right) + 1_{]t_i,t_{i+1}[}(b(t))b'(t)e_{t_i,b(t)}.$$

With this definition of $u_\varepsilon^t$ we have $\mathcal{U}_{\varepsilon u_\varepsilon^t}X_t = X_{t-\varepsilon}$, $\mathcal{A}_s f(X_t) = \frac{1}{2}1_{[a(t),b(t)]}(s)h(s)f''(X_t)$, and

$$(\mathcal{A}f(X_t), u_0^t)_{L^2(\mathbf{R}_+)} = \frac{1}{2}(-a'(t)h^2(a(t)) + b'(t)h^2(b(t)))\partial^2 f(X_t),$$

hence the pseudo generator of $(X_t)_{t\in\mathbf{R}_+}$ is

$$\mathcal{L}_t = \frac{1}{2}(-a'(t)h^2(a(t)) + b'(t)h^2(b(t)))\partial^2.$$

• In the following example the computation of a conditional expectation is needed in order to calculate $\mathcal{L}_t$. Let $X_t = B_t(B_1 - B_t)$, $t \in [0,1]$. With $u^t = e_{0,t} - e_{t,1}$ we have $\mathcal{U}_{\varepsilon u}X_t = X_{t-\varepsilon}$, and

$$(\mathcal{A}f(X_t), u^t)_{L^2(\mathbf{R}_+)} = \frac{1}{2}\left((B_1 - B_t)^2 - B_t^2\right)f''(X_t).$$

The conditional expectation $E[X^2 \mid XY]$ where $X, Y$ are independent centered gaussian random variables with variances $a^2$, $b^2$ respectively can be be computed as

$$E[X^2 \mid XY] = \frac{a}{b} \mid XY \mid \frac{K_1\left(\frac{|XY|}{ab}\right)}{K_0\left(\frac{|XY|}{ab}\right)},$$

where $K_\nu(x)$ is the modified Bessel function of the second kind and of order $\nu \in \mathbb{N}$, hence

$$\mathcal{L}_t = \frac{1}{2} \mid x \mid \left(\sqrt{\frac{1-t}{t}} - \sqrt{\frac{t}{1-t}}\right)\frac{K_1\left(\mid x \mid / \sqrt{t(1-t)}\right)}{K_0\left(\mid x \mid / \sqrt{t(1-t)}\right)}\partial_x^2.$$

To end this section we discuss possible directions for the extension of the above results. The first problem that occurs is that the operator $\nabla^\ominus$ requires the Fock kernels it acts on to be smooth. Hence a construction involving distributions can be useful to generalize the theory, and the Hida calculus seems to be a natural tool here because the operator $\nabla^\ominus$ becomes continuous in Sobolev spaces of Fock kernels as it acts by derivation of these kernels. Since the expectation of $\nabla^\ominus$ in the Itô formula is zero,

the construction of pseudo-generators can be done without smoothness assumptions kernels, for example on the space $\mathcal{D}$. In this case the remaining problem is with the proper definition of the operator

$$\mathcal{A}_s = \int_M \left( 1_{\{x \neq 0\}} \nabla_{x,s}^- + \frac{1}{2} 1_{\{x=0\}} \nabla_{0,s}^- \nabla_{0,s}^- \right) \mu(dx),$$

which requires give a meaning to the contraction of a function of two variables. The right definition may consist in taking right limits, as suggests the following formal treatment of classical diffusions. Let $(X_t)_{t \in \mathbf{R}_+}$ be defined by the stochastic differential equation

$$X_t = \int_0^t \sigma(X_s) dB_s, \quad t \in \mathbf{R}_+, \ \sigma \in \mathcal{C}_c^2(\mathbb{R}).$$

Here $X_t \notin \mathcal{D}$, and we indicate how the usual result can be formally recovered. We have $X_t \in Dom(\nabla^-)$, cf. [14], and

$$\nabla_s^- f(X_t) = \left( \int_s^t \nabla_s^- \sigma(X_v) dB_v + 1_{[0,t]}(s) \sigma(X_s) \right) f'(X_t),$$

and for $u > s$,

$$\begin{aligned}
\nabla_u^- \nabla_s^- f(X_t) &= \int_s^t \nabla_u^- \nabla_s^- \sigma(X_v) dB_v f'(X_t) \\
&\quad + \left( \int_s^t \nabla_s^- \sigma(X_v) dB_v + 1_{[0,t]}(s) \sigma(X_s) \right) \times \\
&\qquad \left( \int_u^t \nabla_u^- \sigma(X_v) dB_v + 1_{[0,t]}(u) \sigma(X_u) \right) f''(X_t),
\end{aligned}$$

hence we can define

$$\mathcal{A} f(X_t) = \frac{1}{2} \lim_{s \to t^-} \lim_{u \to s^+} \nabla_u^- \nabla_s^- f(X_t) = \frac{1}{2} \sigma^2(X_t) f''(X_{t+\varepsilon}), \tag{29}$$

and obtain $\mathcal{L} = \frac{1}{2} \sigma^2(x) \partial_x^2$.

# 7 Construction of the Skorohod integral

The purpose of this section is to construct the two different Skorohod integrals induced by the operators $\nabla^+$ and $\nabla^\oplus$ as adjoints of gradient operators, depending on the type (space or time) of perturbation chosen for the Poisson process. Given the identification between $\Gamma(L^2(M \times \mathbb{R}_+)) \otimes L^2(M \times \mathbb{R}_+)$ and $L^2(\Omega) \otimes L^2(M \times \mathbb{R}_+)$ via multiple stochastic integrals, the following natural definition of adaptedness in $\Gamma(L^2(\mathbb{R}_+, \mathcal{K})) \otimes L^2(\mathbb{R}_+, \mathcal{K})$ coincides with the definition of adaptedness with respect to the filtration $(\mathcal{F}_t)_{t \in \mathbf{R}_+}$ generated by $X$.

**Definition 9** *A process $(u_t)_{t \in \mathbb{R}_+}$ with values in $\Gamma(L^2(\mathbb{R}_+, \mathcal{K})) \otimes \mathcal{K}$ is said to be adapted if*

$$(u_t, h_n)_{\Gamma(L^2(\mathbb{R}_+, \mathcal{K})) \otimes \mathcal{K}} = 0,$$

*whenever $h_n \in L^2([t, \infty[, \mathcal{K})^{\circ n} \otimes \mathcal{K}$, $n \geq 0$, $t \in \mathbb{R}_+$.*

The set of simple processes defined below is dense in $\Gamma(L^2(\mathbb{R}_+, \mathcal{K})) \otimes L^2(\mathbb{R}_+, \mathcal{K}) \simeq L^2(\Omega) \otimes L^2(M \times \mathbb{R}_+)$.

**Definition 10** *We say that $u \in \Gamma(L^2(\mathbb{R}_+, \mathcal{K})) \otimes L^2(\mathbb{R}_+, \mathcal{K})$ is a simple process if it is written as*

$$\sum_{i=1}^{i=n} F_i u_i,$$

*where $u_1, \ldots, u_n \in \mathcal{C}_c(M \times \mathbb{R}_+)$, and $F_1, \ldots, F_n \in \mathcal{S}$, $n \geq 1$. This set of processes is denoted by $\mathcal{U}$.*

The first definition of the Skorohod integral uses the operator $\nabla^+$, and is the most frequently used on the Wiener space, cf. [14], [15]. Let $\pi : L^2(M \times \mathbb{R}_+) \to L^2(M^* \times \mathbb{R})$ denote the canonical projection.

**Proposition 10** *Let $u \in \mathcal{U}$ be a simple process in $L^2(\Omega, P) \otimes L^2(M \times \mathbb{R}_+)$. We have*

$$
\begin{aligned}
\nabla^+(u) &= \int_{M \times \mathbb{R}_+} u(x, t) X(dx, dt) - \int_{M \times \mathbb{R}_+} \nabla^-_{x,t} u(x, t) \mu(dx) dt \\
&\quad - \nabla^+ \left( \pi \left( \nabla^-_\cdot u(\cdot) \right) \right),
\end{aligned}
\tag{30}
$$

*and if $u \in L^2(\Omega) \otimes L^2(M \times \mathbb{R}_+)$ is $(\mathcal{F}_t)$-adapted, then*

$$\nabla^+(u) = \int_{M \times \mathbb{R}_+} u(x, t) \tilde{X}(dx, dt).$$

*Proof.* We work for a process of the form $u = Fh$ and we use Prop. 1 to express the multiplication of $F \in \mathcal{S}$ by $I_1(h)$ as a sum of three terms including $\nabla^+(F \otimes h)$, and obtain (30). This relation is then extended to $\mathcal{U}$ by linearity. The definition of adaptedness implies that the correction terms vanish in the adapted case.
$\square$

The second notion of Skorohod integral uses the operator $\nabla^\oplus$ and the splitting of $d\Gamma(g)$ into $a_g^\ominus$ and $a_g^\oplus$ in Prop. 2. If $u \in \mathcal{U}$ is a simple process written as $u = \sum_{i=1}^{i=n} F_i u_i$

we define the operator $\tilde{D}_u$ on $\mathcal{S}$ by time changes and infinitesimal shifts of Brownian motion from Prop. 4 and Prop. 6 as

$$\tilde{D}_u F = (\nabla^- F, u)_{L^2(M \times \mathbf{R}_+)} + \sum_{i=1}^{i=n} F_i a_{\pi u_i}^{\ominus} F, \quad F \in \mathcal{S}.$$

The operators $\tilde{D}$ and $\tilde{\delta} = \nabla^+ + \nabla^{\oplus} \circ \pi$ are adjoints in the following sense. We have

$$
\begin{aligned}
E[\tilde{D}_u F] &= < \nabla^- F, u >_{\Gamma(\mathcal{H}) \otimes \mathcal{H}} + E[\sum_{i=1}^{i=n} F_i a_{\pi u_i}^{\ominus} F] \\
&= < F, \nabla^+(u) >_{\Gamma(\mathcal{H})} + E[F \sum_{i=1}^{i=n} a_{\pi u_i}^{\oplus} F_i] \\
&= < F, \nabla^+(u) >_{\Gamma(\mathcal{H})} + E[F \sum_{i=1}^{i=n} \nabla^{\oplus}(\pi u_i F_i)] \\
&= < F, \nabla^+(u) + \nabla^{\oplus}(\pi u) >_{\Gamma(\mathcal{H})} = < F, \tilde{\delta}(u) >_{\Gamma(\mathcal{H})}, \quad u \in \mathcal{U}, \ F \in \mathcal{S}.
\end{aligned}
$$

For $u \in \mathcal{U}$ with $u = \sum_{i=1}^{i=n} F_i u_i$, we define $trace(\tilde{D}u)$ as

$$trace(\tilde{D}u) = \int_0^{\infty} \nabla_s^- u_s ds + \sum_{i=1}^{i=n} a_{\pi u_i}^{\ominus} F_i.$$

**Proposition 11** *If $u \in \mathcal{U}$ is a simple process in $L^2(\Omega) \otimes L^2(M \times \mathbf{R}_+)$, then*

$$\tilde{\delta}(u) = \int_{M \times \mathbf{R}_+} u(x,t) X(dx, dt) - trace(\tilde{D}u).$$

*If moreover $u \in L^2(\Omega) \otimes L^2(M \times \mathbf{R}_+)$ is $(\mathcal{F}_t)$-adapted, then*

$$\tilde{\delta}(u) = \nabla^+(u) = \int_{M \times \mathbf{R}_+} u(x,t) \tilde{X}(dx, dt).$$

*Proof.* For $u$ of the form $u = Fh$ we use Prop. 10 and the decomposition of the number operator that follows from (9), (10) and (11):

$$\nabla^+(\pi \nabla_\cdot^- u_\cdot) = \nabla^+(\pi h . \nabla_\cdot^- F) = a_{\pi h}^{\ominus} F + a_{\pi h}^{\oplus} F = a_{\pi h}^{\ominus} F + \nabla^{\oplus}(\pi u).$$

Finally we use the fact that $\nabla^{\oplus}$ vanishes on adapted processes from its definition (7).

$\square$

# 8 Clark formula

In this section we extend the Clark formula, cf. [7], [25], to the case of Lévy processes.

**Proposition 12** *For $F \in L^2(\Omega)$, we have*

$$F = E[F] + \int_{M \times \mathbf{R}_+} E[\nabla_{x,t}^- F \mid \mathcal{F}_t] \tilde{X}(dx, dt).$$

*Proof.* Let $\Delta_n = \{((x_1, t_1), ..., (x_n, t_n)) \in (M \times \mathbb{R}_+)^n \ : \ t_1 < \cdots < t_n\}$. We have for $F \in \mathcal{S}$:

$$
\begin{aligned}
F &= E[F] + \sum_{n \geq 1} I_n(f_n 1_{\Delta_n}) \\
&= E[F] + \sum_{n \geq 1} \int_{M \times \mathbf{R}_+} I_{n-1}(f_n(\cdot, x, t) 1_{\Delta_n}(\cdot, x, t)) \tilde{X}(dx, dt) \\
&= E[F] + \int_{M \times \mathbf{R}_+} \sum_{n \geq 0} E[I_n(f_{n+1}(\cdot, x, t) 1_{\Delta_n}) \mid \mathcal{F}_t] \tilde{X}(dx, dt) \\
&= E[F] + \int_{M \times \mathbf{R}_+} E[\nabla_{x,t}^- F \mid \mathcal{F}_t] \tilde{X}(dx, dt)
\end{aligned}
$$

The extension of this statement to $F \in L^2(\Omega)$ is a consequence of the fact that the adapted projection of $\nabla^- F$ extends to a continuous operator from $L^2(\Omega)$ into the space of adapted processes in $L^2(\Omega) \otimes L^2(M \times \mathbf{R}_+)$. For $F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathcal{S}$ and $u = \sum_{n=0}^{\infty} I_n(u_{n+1}) \in \mathcal{U}$ with $u_{n+1} \in L^2(\mathbb{R}_+, \mathcal{K})^{\circ n} \otimes L^2(\mathbb{R}_+, \mathcal{K})$, $n \in \mathbb{N}$, we can extend a classical argument:

$$
\begin{aligned}
&\left| E\left[ \int_{M \times \mathbf{R}_+} u(x, t) E[\nabla_{x,t}^- F \mid \mathcal{F}_t] \mu(dx) dt \right] \right| \\
&\leq \sum_{n=0}^{\infty} (n+1)! \left| \int_{M \times \mathbf{R}_+} (f_{n+1}(\cdot, x, t) 1_{[0,t]}(\cdot), u_{n+1}(\cdot, t))_{L^2(\mathbf{R}_+, \mathcal{K})^{\otimes n}} \mu(dx) dt \right| \\
&\leq \sum_{n=0}^{\infty} n! \sqrt{n+1} \parallel f_{n+1} \parallel_{L^2(\mathbf{R}_+, \mathcal{K})^{\otimes (n+1)}} \parallel u_{n+1} \parallel_{L^2(\mathbf{R}_+, \mathcal{K})^{\otimes (n+1)}} \\
&\leq \left( \sum_{n=0}^{\infty} n! \parallel f_n \parallel_{L^2(\mathbf{R}_+, \mathcal{K})^{\otimes n}}^2 \sum_{n=0}^{\infty} n! \parallel u_{n+1} \parallel_{L^2(\mathbf{R}_+, \mathcal{K})^{\otimes (n+1)}}^2 \right)^{1/2} \\
&\leq \parallel F \parallel_{L^2(\Omega)} \parallel u \parallel_{L^2(\Omega) \otimes L^2(M \times \mathbf{R}_+)} .
\end{aligned}
$$

$\square$

26

# References

[1] S. Albeverio, Yu. G. Kondratiev, and M. Röckner. Differential geometry of Poisson spaces. *Comptes Rendus de l'Académie des Sciences de Paris, Série I*, 323:1129–1134, 1996.

[2] D. Applebaum. Spectral families of quantum stochastic integrals. *Communications in Mathematical Physics*, 170:607–628, 1995.

[3] D. Applebaum. Covariant Poisson fields in Fock space. To appear in K.R. Parthasarathy's 60th birthday volume, 1996.

[4] R.J. Bass and M. Cranston. Malliavin calculus for jump processes and applications to local time. *Annals of Probability*, 17:507–622, 1984.

[5] K. Bichteler, J.B. Gravereaux, and J. Jacod. *Malliavin Calculus for Processes with Jumps*. Stochastics Monographs. Gordon and Breach, 1987.

[6] E. Carlen and E. Pardoux. Differential calculus and integration by parts on Poisson space. In *Stochastics, Algebra and Analysis in Classical and Quantum Dynamics*, pages 63–73. Kluwer, 1990.

[7] J.M.C. Clark. The representation of functionals of Brownian motion by stochastic integrals. *Annals of Mathematical Statistics*, 41:1281–1295, 1970.

[8] A. Dermoune, P. Krée, and L. Wu. Calcul stochastique non adapté par rapport à la mesure de Poisson. In *Séminaire de Probabilités XXII*, volume 1321 of *Lecture Notes in Mathematics*. Springer Verlag, 1988.

[9] L. Gross. Abstract Wiener spaces. In *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley, 1967. Univ. of California Press.

[10] T. Hida, H. H. Kuo, J. Potthoff, and L. Streit. *White Noise, an infinite dimensional calculus*. Kluwer Academic Publishers, Dordrecht, 1993.

[11] J.A. León and C. Tudor. A chaos approach to the anticipating calculus for the Poisson process. Preprint, 1996.

[12] P. Malliavin. Stochastic calculus of variations and hypoelliptic operators. In *Intern. Symp. SDE. Kyoto*, pages 195–253, Tokyo, 1976. Kinokumiya.

[13] P.A. Meyer. *Quantum Probability for Probabilists*, volume 1538 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin/New York, 1993.

[14] D. Nualart. *The Malliavin Calculus and Related Topics*. Probability and its Applications. Springer-Verlag, Berlin/New York, 1995.

[15] D. Nualart and E. Pardoux. Stochastic calculus with anticipative integrands. *Probability Theory and Related Fields*, 78:535–582, 1988.

[16] D. Nualart and J. Vives. Anticipative calculus for the Poisson process based on the Fock space. In J. Azéma, P.A. Meyer, and M. Yor, editors, *Séminaire de Probabilités de l'Université de Strasbourg XXIV*, volume 1426 of *Lecture Notes in Mathematics*, pages 154–165. Springer Verlag, 1990.

[17] K.R. Parthasarathy. *An Introduction to Quantum Stochastic Calculus*. Birkäuser, 1992.

[18] J. Picard. On the existence of smooth densities for jump processes. *Probability Theory and Related Fields*, 105:481–511, 1996.

[19] N. Privault. Chaotic and variational calculus in discrete and continuous time for the Poisson process. *Stochastics and Stochastics Reports*, 51:83–109, 1994.

[20] N. Privault. Stochastic calculus of variations for martingales. In Ian M. Davis, Aubrey Truman, and K. David Elworthy, editors, *Proceedings of the Fifth Gregynog Symposium*, Stochastic Analysis and Applications, pages 385–400, Gregynog, 1995. World Scientific.

[21] N. Privault. Calculus on Fock space and a quantum non-adapted Itô formula. *Comptes Rendus de l'Académie des Sciences de Paris, Série I*, 323:927–932, 1996.

[22] N. Privault. A calculus on Fock space and its probabilistic interpretations. Prépublication de l'Equipe d'Analyse et Probabilités, Université d'Evry, 1996.

[23] F. Russo and P. Vallois. Product of two multiple stochastic integrals with respect to a normal martingale. Preprint, 1996.

[24] D. Surgailis. On multiple Poisson stochastic integrals and associated Markov semigroups. *Probability and Mathematical Statistics*, 3:217–239, 1984.

[25] A.S. Üstünel. Representation of the distributions on Wiener space and stochastic calculus of variations. *J. Funct. Anal.*, 70:126–129, 1987.

[26] A.S. Üstünel. The Itô formula for anticipative processes with non-monotonous time via the Malliavin calculus. *Probability Theory and Related Fields*, 79:249–269, 1988.

[27] A.S. Üstünel. *An Introduction to Analysis on Wiener Space*, volume 1610 of *Lecture Notes in Mathematics*. Springer Verlag, Berlin/New York, 1995.