OPTIMIZATION OF CONSUMPTION WITH LABOR INCOME

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Abstract

We present in a closed form the solution of a portfolio optimization problem for an economic agent endowed with a stochastic insurable stream, under a liquidity constraint over the time interval $[0, T]$. Generally, the existence of labor income complicates the agent's decisions. Moreover, in the real world the economic agents are restricted in their ability to borrow against their future labor income. We deal with this kind of liquidity constraint following the lines of American option valuation.

1 Introduction

This paper studies the individual's optimal consumption and portfolio strategies for an economic agent who receives or pays a stochastic wage income throughout his lifetime.

The economic agents in the "real world" are restricted in their ability to borrow against their future labor income and the liquidity constraints and/or the presence of an income risk generally change the agent's tolerance to the risk in the financial market.

In a classical framework of consumption-investment problem, the terminal wealth is constrainted to be non-negative. In many cases, when the investor is not endowed with an income process, this assumption suffices to guarantee that the optimally invested wealth will never reach zero before terminal time, and the non-negativity constraint on current wealth is not binding as it was proved in Merton (1971), Cox-Huang (1989), Karatzas, Lehoczky, Sethi and Shreve (1986), Karatzas, Lehoczky, Shreve and Xu (1991).

When the investor receives an income, the non-negative terminal wealth constraint does not imply that the optimal non-constrained current wealth is non-negative: this problem is studied in some special case in the seminal paper of Merton (1971) and in a general setting in Karatzas, Lakner, Lehoczky and Shreve (1991) and Koo (1995). However these authors ignore the effect of non-negative wealth constraint over the period of trading and allow the individual to capitalize the lifetime flow of wage income at some rate of interest which lead them to treat the capitalized value as an addition to the current stock of wealth. Bardhan (1993) studies the problem of stochastic income in an incomplete financial market by means of equivalent martingale measure and identify a necessary and sufficient budget-feasibility condition that determines whether a consumption investment plan can be financed relative to the investor endowment; however, the income over the period allows the individual to overcome an initial negative capital and finance a consumption since the author does not impose non-negative wealth constraint. We will refer to the case without liquidity constraint as to the free case, the free wealth being its optimal solution.

We impose here the constraint on the investor's wealth to be non-negative over the lifetime interval. This prohibits the agent from selling his wage income in the financial market. As it is well known, this constraint implies that perfect hedging is not possible in general. The same problem is solved by Cuoco (1995) who proves the existence of an optimal strategy, using martingales techniques and working directly on the primal problem. However, he does not obtain closed-form solutions. This problem was also studied in He and Pagès (1993) via an application of duality theory. These authors transform the problem into an unconstrained dual shadow prices problem and establish the existence of an optimal strategy for the primal problem. In their approach, when the securities prices processes follow a Markov-diffusion process, the shadow price problem can be solved by dynamic programming: the Bellman equation corresponding to the dual problem is a linear PDE subject to a free boundary condition. They solve this equation when the labor income is a function of the asset price. Duffie, Fleming, Soner and Zariphopoulou (1996) solved the same kind of problem, when the income stream is assumed to be uninsurable, using the point of view of dynamic programming: in particular, they give a "quasi-explicit" solution of the H.J.B. equation associated with an HARA utility function and deterministic coefficients. However, the assumption that the market is incomplete (the labor income cannot be duplicated by a portfolio) is essential. As a consequence of their study, they obtain that at zero wealth, a fixed fraction of wealth is consumed, the remainder being saved in the riskless asset.

In this paper, which is a new draft of an unpublished one, we define a dominance in terms of excess consumption over the period. Financing a consumption c and a terminal wealth ζ is achieved by a portfolio which allows a consumption greater than c and a terminal wealth equal to ζ. The aim of this paper is to give, under the assumption that the financial market is complete, and with some additional mild conditions, a quasi-closed solution for the constrained problem, i.e., the problem which takes into account the liquidity constraint. We first take into account the risk aversion of the agent by identifying a family of target consumption plans, which are the optimal consumption plans for the free problem before fitting an initial wealth. The first main result is a "separation" theorem: in the constrained case, the optimal solution consists to invest a part of the wealth in the unconstrained strategy and spend the remainder to finance an American Put written on the free wealth, in order to provide an insurance against the constraint. The second main result is that there is no needless consumption for the optimal constraint solution.

In a first part, we define the model and give a static characterization of the feasible consumption plans which are financable from an initial wealth y less than or equal to x; when the liquidity constraint $X_t \geq 0$ is required at any time $t \in [0, T]$, and more precisely at any stopping time $\tau \in [0, T]$, an infinite number of budget constraints -associated with the different horizons τ- have to be satisfied (He and Pagès (1993)). The connections with optimal stopping times problem and American options are detailled. Let us remark that the liquidity constraint can be on the form $X \geq U$ where U is the benchmark portfolio process.

In a second step, we recall results in the case where the liquidity constraint does not hold (free-problem); this problem was completely solved in Cox and Huang (1989) and Karatzas, Lehocky, and Shreve (1987) by mean of a family of consumption plan $(c^f(\nu), \zeta^f(\nu))$ which is optimal for the dual problem. The optimal solution of the primal problem is obtained by fitting the Kuhn-Tucker multiplier ν such that the budget constraint is saturated.

Then, we introduce the dual problem, in which the budget constraint is linearized by mean of the Kuhn-Tucker multipliers. We solve this dual problem, using some extended results on the related singular control problem. In particular, we show that the derivative of the dual value function is the value function of an optimal stopping times problem. The optimal solution of the dual problem is then constructed from the inverse of an optimal stopping times family. Necessary optimality conditions are used in a following step to show that the optimal solution for the dual problem leads to a solution for the primal problem, choosing a Kuhn-Tucker multiplier which saturates the constraint.

We introduce the stochastic boundary associated with the optimal stopping problem, and show that the optimal solution is obtained by forcing the shadow price of the free problem to stay below this boundary. In terms of the primal problem, when the shadow price reaches the boundary, the optimal wealth is equal to 0 and the constraint is active. However, the agent may consume a certain part of the income. Despite the fact that the wealth is equal to zero along this boundary, this boundary is not an absorbing one. These properties appear clearly in the particular case when all the parameters of the market are constant, the insurable income stream evolves according to a geometrical Brownian motion and the utility functions are HARA utility functions. Using some classical results about American put options on paying dividend assets in a Black and Scholes framework, we describe the solution in the last section. Moreover, when the horizon is infinite, we exhibit an explicit solution.

2 The Optimization Problem

We consider an investment-consumption problem for an agent who receives (or pays) a stochastic income stream and is submitted to liquidity constraints.

2.1 Statement of the Problem

The Model

We begin with the typical setup for continuous-time asset pricing. There are $n+1$ financial assets which the agent can buy and/or sell without incurring any cost of trading. One of them is a non risky asset (the money market instrument) with price per unit $S_0(t)$ governed by the equation:

$$dS_0(t) \;=\; S_0(t)\, r_t\, dt, \quad S_0(0) = 1$$

where r_t is the short rate. In addition to the bond, n risky securities (the stocks) are continuously traded. The price process $(S_i(t), t \geq 0)$ for one share of the i^{th} stock is modeled by the linear stochastic differential equation:

$$dS_i(t) \;=\; S_i(t)[\, b_i(t)\, dt \;+\; \sum_{j=1}^{n} \sigma^{i,j}(t)\, dW_t^j\,]$$

where $W = (W^1, \ldots, W^n)^*$ is a standard Brownian motion on $\mathbb{R}^n$, defined on a probability space $(\Omega, \mathcal{F}, P)$. The information structure is given by the filtration $(\mathcal{F}_t\,;\, 0 \leq t \leq T)$ generated by the Brownian motion W and augmented. Throughout this paper, the usual following conditions are satisfied:

Hypotheses

• The short rate process $(r_t, t \geq 0)$, the column vector of the stock appreciation rates $(b_t = (b_1(t), \ldots, b_n(t))^*, t \geq 0)$ and the volatility matrix $(\sigma_t = (\sigma^{i,j}(t)), t \geq 0)$ are predictable and bounded. For any time t, the matrix σ_t has a full rank.
• The financial market is arbitrage-free and complete; furthermore there exists a predictable and bounded vector $(\theta_t, t \geq 0)$, called the risk premium vector, such that

$$b_t - r_t \mathbf{1} \;=\; \sigma_t\, \theta_t \quad (dP \otimes dt) - a.s.\,,$$

where $\mathbf{1}$ is the vector in $\mathbb{R}^n$ with all components equal to 1.

We denote by T a fixed horizon, and introduce some integrability conditions :
• The set H_2 (resp. H_2^n) consists of the class of all $(\mathcal{F}_t)$-progressively measurable, $\mathbb{R}$-valued, (resp. $\mathbb{R}^n$-valued) processes ϕ such that $E\Big(\int_0^T |\phi_t|^2\, dt\Big) < \infty$. With an

obvious notation, H_2^+ is the subset of non-negative processes in H_2.
• The space of non-negative $\mathcal{F}_T$-measurable random variables ζ such that $E(\zeta^2) < +\infty$ is denoted by L_2^+.

The Individual Consumption problem

We consider an agent whose actions cannot affect market prices, who receives an income at time t at the rate e_t, and chooses at each time $t \in [0, T]$ his consumption rate c_t and the amount $\tilde{\pi}^i(t)$ to be invested in the i^{th}-stock ($i = 1, \dots, n$), using a self-financing strategy. The agent's financial wealth $(X_t, t \geq 0)$ evolves according to the equation:

$$dX_t = r_t X_t \, dt - (c_t - e_t) \, dt + \tilde{\pi}_t^* \sigma_t \, (dW_t + \theta_t dt). \tag{2.1}$$

We shall refer to the wealth process solution of (2.1) with the initial endowment $X_0 = x$ as to $(X_t^{x,c,\tilde{\pi}}, t \geq 0)$.

We assume that the income process $(e_t, t \geq 0)$ is spanned by the marketed assets and therefore is not a source of new uncertainty. In particular, e is $(\mathcal{F}_t)$-adapted. Similar assumptions are introduced in He and Pagès (1993) and more recently in Detemple and Serrat (1996), in opposite to the papers of Duffie et al. (1996) and Koo (1995) where the existence of a specific noise in the dynamics of the labor income is a key argument to solve the problem. In what follows, the income process $(e_t, t \geq 0)$ is assumed to be in H_2.

In this paper, we suppose that the **liquidity constraint** $X_t \geq 0$ is to be satisfied at any time t before T. The existence of such a constraint implies that the agent cannot borrow against future labor income.

A triple (initial wealth-consumption-investment) $(x, c, \tilde{\pi}) \in I\!R^+ \times H_2^+ \times H_2^n$ is said to be **feasible** if the liquidity constraint $(X_t^{x,c,\tilde{\pi}} \geq 0, t \in [0, T])$ is satisfied almost surely by the current wealth, solution of the equation (2.1). The set $\mathcal{A}^+(x)$ of x-feasible consumption profiles consists of consumption processes c in H_2^+ associated with a portfolio $\tilde{\pi}$ in H_2^n such that the triple $(x, c, \tilde{\pi})$ is feasible.

Remarks: **a.** When $(e_t \geq 0, \forall t \geq 0)$, this stochastic flow may be interpreted as a labor wage. When $(e_t \leq 0, \forall t \geq 0)$, the problem may be interpreted as a constraint on the consumption, by redefining the consumption in excess of the given income process e: $\tilde{c}_t = c_t - e_t \geq -e_t$.
b. If we fit the consumption rate to the income rate such that $(c_t \geq (e_t \vee 0), t \in [0, T])$, the problem is reduced to a portfolio management problem, and it is well-known that the assumption of a non-negative terminal wealth implies the liquidity constraint.
c. Liquidity constraints, of course, can take a different form, e.g., there are risky consumer loans available to the agent when the wealth of the agent is small or negative, but in general the loan amount is limited by $(M_t, t \geq 0)$ (supposed to be an Itô process). Such constraint may be easily modeled in our framework, by replacing the wealth by the cushion $\tilde{X}_t = X_t - M_t$. In

the insurance portfolio problem the wealth is required to satisfy $X_t \geq U_t, \forall t$, where U is the benchmark portfolio. This problem can be studied by considering the process $X - U$.

Utility functions

The agent's preferences over consumption and wealth profiles are given by time-additive utility functions u for consumption and g for the terminal wealth. We recall some classical definitions and properties of utility functions.

- A utility function $U :]0,\infty[\to I\!R$ is a strictly increasing, strictly concave, continuously differentiable function which satisfies $U'(\infty) \stackrel{def}{=} \lim_{x\to\infty} U'(x) = 0$, and eventually the Inada condition $U'(0) \stackrel{def}{=} \lim_{x\to 0} U'(x) = \infty$. If the Inada condition does not holds, we suppose that $U(0)$ is finite.

- We shall denote by $\tilde{U}$ the convex conjugate of U, defined on $I\!R_+^*$ as

$$\tilde{U}(y) = \sup_{x>0}(U(x) - xy)\,. \tag{2.2}$$

  The function $\tilde{U}$ is strictly convex and strictly decreasing on $]0, U'(0)[$. If $y > U'(0)$, the supremum being relative to non-negative values of x we obtain $\tilde{U}(y) = U(0)$. The function U' admits a strictly decreasing inverse, defined on $]0, U'(0)[$ by $\mathcal{I}(y) = \inf\{x \geq 0 \,|\, U'(x) \leq y\}$. We set $\mathcal{I}(y) = 0$ for $y \geq U'(0)$. It is well known that $\mathcal{I}(y) = -\tilde{U}'(y) \geq 0$ on $]0, U'(0)[$ and this equality extends to the whole interval $]0, +\infty[$ since $\mathcal{I}(y) = -\tilde{U}'(y) = 0$ for $y \geq U'(0)$. Furtheremore, $\lim_{y\to\infty} -\tilde{U}'(y) = 0$ and $\lim_{y\to 0} -\tilde{U}'(0) = \infty$.

- The following relation holds for each $y \in]0, U'(0)[$

$$U(-\tilde{U}'(y)) + y\tilde{U}'(y) = \tilde{U}(y)$$

  which implies that, for $y \in]0, U'(0)[$ the supremum in (2.2) is achieved for $x = \mathcal{I}(y)$. Otherwise, i.e., if $y \geq U'(0)$, the maximum is achieved for $x = 0 = -\tilde{U}'(y)$.

We shall consider throughout a map $u :]0,\infty[\times[0,T]\times\Omega \to I\!R$ such that for any given $(t,\omega) \in [0,T]\times\Omega$ the function $u(.,t,\omega)$ is a utility function[1] and, for any $x \in I\!R_+^*$ the process $u(x,\cdot)$ is $(\mathcal{F}_t)$-adapted. We denote by u' the derivative of u with respect to the first argument, and by $\tilde{u}(\cdot,t)$ the convex conjugate of $u(\cdot,t)$. The function $g : I\!R_+^* \times \Omega \to I\!R$ is such that for any $x > 0$, the random variable $g(x,\cdot)$ is $\mathcal{F}_T$-mesurable and $g(\cdot,\omega)$ is a utility function. The HARA functions

[1] We consider the case where $(u(x,t), t \geq 0)$ is an adapted process which allows us to deal with the case of insurance portfolio and cushion where the utility process is on the form $(u(e_t + c), t \geq 0)$.

$e^{-\beta t}\dfrac{c^{1-\gamma}}{1-\gamma}$ (with $-\tilde{u}(y,t) = (ye^{\beta t})^{-1}$) and the logarithmic function are classical examples of deterministic utility functions. The generalized HARA case, where $u(c,t) = \dfrac{\gamma e^{-\beta t}}{1-\gamma}\left(\dfrac{ac}{\gamma} + b\right)^{1-\gamma}$ is studied in Cox and Huang (1989), Cuoco (1996) and other authors.

Remark: The role of the Inada condition is discussed in Sethi and Taksar (1988), Sethi (1995), Teplà (1996) and in the forthcoming book of Karatzas and Shreve (1996). This assumption is not needed in the main part of our paper.

Optimisation criteria

Given an initial endowment $x \geq 0$ and an income stream $(e_t, t \leq T)$, an investor wishes to choose a consumption profile and an investment policy $\tilde{\pi}$ so as to maximize his total expected utility from consumption over the period and expected utility of investment at the end of the period $[0,T]$,

$$E\Big(\int_0^T u(c_s, s)\, ds + \, g(X_T^{x,c,\tilde{\pi}})\Big)$$

using feasible policies.

2.2 Liquidity constraint

As it is now well known, the liquidity constraint $X_t^{x,c,\tilde{\pi}} \geq 0$ over the time interval $[0,T]$ can be formulate as a budget constraint which involves the value ζ of the terminal wealth.

Hedgeable consumption plan, free-value

When there is no income ($e \equiv 0$), it is well known that in a complete market, for a given consumption plan $(c,\zeta) \in H_2^+ \times L_2^+$, there exists a portfolio strategy $\tilde{\pi}$ and an initial endowment x such that the wealth process $X^{x,c,\tilde{\pi}}$ satisfies the liquidity constraint $X^{x,c,\tilde{\pi}} \geq 0$ (or equivalently, there exists x such that $c \in \mathcal{A}^+(x)$) and the terminal condition $X_T^{x,c,\tilde{\pi}} = \zeta$. In this case, the non-negative terminal wealth condition implies the liquidity constraint.
Let us now allow the presence of a stochastic income e. For notational simplicity, we denote by $\pi_t \overset{def}{=} \tilde{\pi}_t \sigma_t^*$ the stochastic part of a self-financing portfolio. Since the market is complete, all consumption plans $(c,\zeta) \in H_2^+ \times L_2^+$ are **hedgeable**, i.e., there exists a pair of processes $(X^{c,\zeta}, \pi) \in H_2 \times H_2^n$ such that,

$$\begin{aligned} dX_t^{c,\zeta} &= r_t X_t^{c,\zeta}\, dt - (c_t - e_t)\, dt + \pi_t^*\, (dW_t + \theta_t dt), \\ X_T^{c,\zeta} &= \zeta. \end{aligned} \tag{2.3}$$

Let us emphasize that an hedgeable consumption plan does not necessary lead to a positive wealth; if the wealth process is non-negative over the time interval $[0,T]$, we call it the **free-price** process of the pair (c,ζ) and the pair (c,ζ) is said to be **feasible**.

The wealth $X_t^{c,\zeta}$, called hereafter the $(\mathbf{c},\zeta)$-**free-value** at time t, is the t-time value of a paying dividend rate $(e-c)$ contingent claim ζ. From the arbitrage pricing theory in a complete market and from the square integrability of the consumption plan (c,ζ), we deduce the classical closed formula for the free-value at time t,

$$X_t^{c,\zeta} = E\Big(\int_t^T H_s^t(c_s - e_s)\, ds + H_T^t\, \zeta \,|\mathcal{F}_t\Big) \tag{2.4}$$

where $(H_s^t, s \geq t)$ is the shadow state-price process, also called the deflator process, defined by the following forward equation,

$$dH_s^t = -H_s^t\,(r_s\, ds + \theta_s dW_s), \quad s \geq t\,, \quad H_t^t = 1. \tag{2.5}$$

We set $H_s = H_s^0$ and introduce the present value of the future labor income

$$I_0 \overset{def}{=} E\Big(\int_0^T H_t e_t dt\Big)\,.$$

The budget constraint is stated as a constraint on the consumption plans

$$E\Big(\int_0^T H_s c_s\, ds + H_T\, \zeta\,\Big) \leq x + I_0 \tag{2.6}$$

Let us remark that the investment-consumption free problem when an income is payed is in fact a classical optimisation problem, where the initial wealth x is enlarged from the present value of the future income at time 0.

The set $\mathcal{A}(x)$ of **x-hedgeable** consumption plans, i.e., which are financable from an initial wealth less or equal to x, consists of consumption plans (c,ζ) in $H_2^+ \times L_2^+$ such that the budget constraint (2.6) holds.

Remarks : **a.** From the formula (2.4) follows the fact that, if the terminal wealth is non-negative and if the income process is non-positive, the free-value is non-negative and therefore is the free-price. However, the x-feasible consumption plans set may happened to be empty.
b. In an infinite horizon setting, there is only a consumption utility function, and we shall use a budget constraint on the form

$$E\Big(\int_0^\infty H_s|c_s - e_s|\, ds\Big) < \infty\,, \quad E\Big(\int_0^\infty H_s(c_s - e_s)\, ds\Big) \leq x\,. \tag{2.7}$$

Superstrategies and fair value

Let us now study the liquidity constraint if the agent receives a marketed stochastic income $(e_t, t \leq T)$. In the case $e \geq 0$, if the individual is allowed to capitalize the lifetime flow of the wage income and treats the capitalized value as an addition to the current stock of wealth, the problem is easily solved (we recall the main results below, in the section devoted to the free case), but the current optimal wealth may be negative [2]. At this stage, we can face up two ways :

- The first is to characterize the feasible consumption plans. This difficult way is studied in Saada (1994).
- The second is to introduce superstrategies which involve a cost process to cover the "uninsurable" liquidity risk induced by the liquidity constraint. We present now this method.

We give a less restricting definition for the value of a portfolio, which introduces an additional flow K in order to supply the liquidity risk; the process K captures the possibility of free disposal of wealth: the agent is allowed not to reinvest some of his wealth if he chooses to do so. We shall refer to the triple (X, π, K) as a $(\mathbf{c}, \zeta)$-**superstrategy** if it satisfies

$$\begin{cases} dX_t^{K,\pi} &= r_t X_t^{K,\pi}\,dt - dK_t - (c_t - e_t)\,dt + \pi_t(dW_t + \theta_t\,dt)\ ; \quad X_T^{K,\pi} = \zeta\,, \\ X_t^{K,\pi} &\geq 0;\ \forall t \leq T\,, \end{cases} \tag{2.8}$$

where $(K_t, t \geq 0)$ belongs to $\mathcal{K}$, the set of adapted right-continuous non-decreasing processes.

Remarks: **a.** We can give the following equivalent formulations of this definition: there exists K such that the triple (X, π, K) is a (c, ζ) super-strategy

- if there exists a wealth process satisfying (2.3) such that $X \geq 0$ and $X_T \geq \zeta$: at the end of the period, there is excess wealth left over;

- or if there exists $\tilde{c}, \tilde{c} \geq c$ such that $\tilde{c}$ is feasible.

b. By using the deflator process, we have $X_t = E(\int_t^T H_s^t(c_s - e_s)\,ds + \int_t^T H_s^t\,dK_s + H_T^t\zeta\,|\mathcal{F}_t)$ where $\int_t^T H_s^t\,dK_s \geq 0$. The cost process $(\int_0^t H_s\,dK_s, t \geq 0)$ finances the undhedgeable constraint of the consumption plan (c, ζ). There is some needless consumption.

c. In Cuoco (1996), the optimisation problem is solved among the superstrategies.

For a given consumption plan there exists many superstrategies, and we are interested in a characterization of a minimal one. Such minimal supertrategies are extensively used in the arbitrage-free pricing of American options (Karatzas and

[2] See He and Pagès (1993) for a counter-example.

Shreve (1987, 1996)), where the American price for a pay-off square integrable process $(\xi_t, t \geq 0)$ is defined as the square integrable minimal superstrategy which dominates $(\xi_t, t \geq 0)$, when no dividends are payed $(c \equiv e)$. The classical relation between American options and optimal stopping times theory leads us to give the following properties of the **fair-value** defined as the minimal investment required to replicate (c, ζ) with a superstrategy. We denote by $\mathcal{T}(t)$ (resp. $\mathcal{T}$) the set of stopping times τ such that $t \leq \tau \leq T$ (resp. $0 \leq \tau \leq T$).

Theorem 2.1 *Let $(c, \zeta) \in H_2^+ \times L_2^+$ be a consumption plan.*
a) *There exists a unique $(Y^{c,\zeta}, K^{c,\zeta}, \pi^{c,\zeta}) \in H_2^+ \times \mathcal{K} \times H_2^n$ such that*

$$\begin{cases} dY_t^{c,\zeta} & = & r_t Y_t^{c,\zeta}\, dt - dK_t^{c,\zeta} - (c_t - e_t)\, dt + \pi_t^{c,\zeta}(dW_t + \theta_t\, dt) \\ Y_T^{c,\zeta} & = & \zeta \\ Y_t^{c,\zeta} & \geq & 0\,,\ \forall t \leq T \end{cases} \tag{2.9}$$

where $K^{c,\zeta} \in \mathcal{K}$ is continuous on $[0,T[$ and satisfies

$$\int_{[0,T[} Y_s^{c,\zeta}\, dK_s^{c,\zeta} = 0.$$

b) *The process $Y^{c,\zeta}$, called the* $(\mathbf{c}, \zeta)$**-fair-value process***, satisfies*

$$Y_t^{c,\zeta} = \operatorname*{ess\ sup}_{\tau \in \mathcal{T}(t)} E\Big(\int_t^\tau H_s^t\,(c_s - e_s)\, ds + H_T^t\, \zeta\, 1\!\!1_{\tau=T} | \mathcal{F}_t\Big)\,, \tag{2.10}$$

its initial value

$$Y_0^{c,\zeta} = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}} E\Big(\int_0^\tau H_s\,(c_s - e_s)\, ds + H_T\, \zeta\, 1\!\!1_{\tau=T}\Big) \tag{2.11}$$

is called the (c, ζ)- fair-value.
The stopping time $\sigma^{c,\zeta} = T \wedge \inf\{s;\, 0 \leq s < T, Y_s^{c,\zeta} = 0\}$ achieves the supremum in (2.11) *and is the smallest optimal stopping time.*

The sketch of the proof is given in the Appendix. Let us remark that the (c, ζ)-fair value process $Y^{c,\zeta}$ is larger than the (c, ζ)-free value process $X^{c,\zeta}$. The positive process $Z^{c,\zeta} = Y^{c,\zeta} - X^{c,\zeta}$ satisfies by linearity the equation (2.9) without dividend $(c \equiv e)$ and zero terminal wealth $(\zeta = 0)$. Moreover, $Z_t^{c,\zeta} \geq \sup[-X_t^{c,\zeta}, 0] = (X_t^{c,\zeta})^-$, *a.s.*. It follows that $Z^{c,\zeta}$ is the price of an American put with exercise price equal to 0, written on the free-value process $X^{c,\zeta}$. Therefore the fair-value of a consumption plan can be splitted into two terms: the first one is the free-value process $X^{c,\zeta}$ of this consumption plan and the second one is an American put option $Z^{c,\zeta}$ with exercise price 0. A minimal superstrategy consists to invest a part of the initial wealth in a free portfolio and to use the remaining part for financing the American put option on this underlying asset.

Remark : In their 1989 paper (p.65), in an no-income situation, Cox-Huang, while studying a case where the non-negativity constraint is binding, have remarked that the agent invests some wealth in a European put option with a zero exercise price to obtain an insurance package.

A consumption plan (c, ζ) is said to be **x-affordable** if the (c, ζ) -fair-value is less than x, i.e. if the following budget constraint holds

$$Y_0^{c,\zeta} \leq x \, . \tag{2.12}$$

We characterize the x-affordable consumption plans by mean of $\mathcal{D}$, the class of adapted, right-continuous, non-increasing processes D such that $D_0 \leq 1$, $D_T = 0$, and we denote by D_T^- the left limit of D at T.

Proposition 2.2 *A consumption plan is x-affordable if and only if*

$$\sup_{D \in \mathcal{D}} E\Big(\int_0^T H_s D_s (c_s - e_s)\, ds + H_T D_T^- \zeta\Big) \leq x. \tag{2.13}$$

PROOF: The proof is given in the appendix. □

3 Resolution of the optimisation problem

As we noticed, the solution of the constrained problem is constructed from the solution of the free problem. Therefore, we need some hypothesis on this solution: in this paper, the following hypothesis[3] prevails:

Hypothesis We suppose that for any $\nu \in I\!R_+^*$, the process $(c_t^f(\nu) \stackrel{def}{=} -\tilde{u}'(\nu H_t, t), t \geq 0)$ belongs to H_2^+ and that $\zeta^f(\nu) \stackrel{def}{=} -\tilde{g}'(\nu H_T)$ is in L_2^+.

The $(c^f(\nu), \zeta^f(\nu))$-free-value process, denoted hereafter by $X^f(\nu)$ will play an important role in our study. This process, given by

$$X_t^f(\nu) \stackrel{def}{=} E\Big(\int_t^T H_s^t \left(-\tilde{u}'(\nu H_s, s) - e_s\right) ds - H_T^t \,\tilde{g}'(\nu H_T) \,|\mathcal{F}_t\Big) \,, \tag{3.1}$$

will be simply called the **free-value process** and its initial value

$$X_0^f(\nu) = E\Big(\int_0^T H_s(-\tilde{u}'(\nu H_s, s) - e_s)\, ds - H_T \,\tilde{g}'(\nu H_T)\Big) \tag{3.2}$$

is simply called the free value.

[3]Refer to He and Pagès' and Cuoco's paper (1996), where the same kind of hypothesis is done, for an expression of this hypothesis in terms of the function u.

3.1 The free case

The **free problem** (where only the terminal wealth is assumed to be non-negative) has been studied by Karatzas, Lakner, Lehoczky and Shreve (1991), Bardhan (1993), Koo (1995) and other authors. We review here those of their results which are of particular importance in the present paper.

The free problem can be restated as the equivalent problem of maximizing expected utility over all x-hedgeable consumption plans (c, ζ)

$$\sup_{(c,\zeta)\in\mathcal{A}(x)} E\Big(\int_0^T u(c_s, s)\, ds + g(\zeta)\Big).$$

It is well known that, under weak assumptions, there exists a complete solution to this optimization problem, constructed from a dual problem.

The optimization problem may be linearized using Lagrangian method and is transformed in

$$\sup_{(c,\zeta)\in H_2^+\times L_2^+} E\Big(\int_0^T u(c_s, s)\, ds + g(\zeta) + \nu(x - \int_0^T H_s(c_s - e_s)\, ds - H_T\,\zeta)\Big)$$

where $\nu \geq 0$ is the Kuhn-Tucker multiplier. Some analytic considerations allow us to invert the supremum and the integral signs. Therefore, the free dual value function is

$$J(\nu) \stackrel{def}{=} E\Big(\int_0^T \sup_{c\geq 0}\,(u(c_s, s) - \nu H_s(c_s - e_s))\, ds + \sup_{\zeta\geq 0}\,(g(\zeta) - \nu H_T\zeta)\Big)$$

and can be written in a form involving the convex conjugate functions

$$J(\nu) = E\Big(\int_0^T (\tilde{u}(\nu H_s, s) + \nu H_s e_s)\, ds + \tilde{g}(\nu H_T)\Big).$$

Theorem 3.1 *The consumption plan $(c^f(\nu), \zeta^f(\nu))$ is optimal for the dual problem.*
For a given endowment $x > -I_0$, there exists a unique $\nu = \nu_x \geq 0$ such that $X_0^f(\nu) = x$.
The consumption plan $(c^f(\nu_x), \zeta^f(\nu_x))$ is optimal for the primal problem among the x-hedgeable consumption plans.

PROOF: From the properties of the utility functions (regularity and concavity), the supremum of $g(\zeta) - \nu H_T\zeta$ is achieved for $\zeta = -\tilde{g}'(\nu H_T)$ and the supremum of $u(c, s) - \nu H_s(c - e_s)$ for $c_s(\nu) = -\tilde{u}'(\nu H_s, s)$. Thanks to the integrability assumptions, this consumption plan is hedgeable, and the current value of this plan is given by (3.1). To match the initial value of this optimal plan with the initial endowment x it is sufficient to choose the multiplier ν such that $x = X_0^f(\nu)$. If the Inada condition holds, the map $\nu \longrightarrow X_0^f(\nu)$ is continuous and strictly decreasing

on $]0,+\infty[$, with $X_0^f(0) = +\infty$ and $X_0^f(\infty) = -I_0 < x$, so such a multiplier ν_x exists and is unique. In particular, if the income is non-negative, the multiplier ν_x exists for all $x \in I\!R_+^*$. If the Inada's condition does not hold, let us introduce $\bar{\nu} = \inf\{\nu, X_0^f(\nu) = -I_0\}$. The map $\nu \longrightarrow X_0^f(\nu)$ is continuous and strictly decreasing on $]0,\bar{\nu}]$, with $X_0^f(0) = +\infty$ and $X_0^f(\bar{\nu}) = -I_0 < x$, and the existence of an unique ν_x follows.
Then it is easy, using convexity inequalities, to show that the consumption plan $(c^f(\nu_x), \zeta^f(\nu_x)) = (-\tilde{u}'(\nu_x H, \cdot), -\tilde{g}'(\nu_x H_T))$ is optimal among the x-hedgeable consumption plans for the primal problem (See Karatzas et al. (1986)). Given the optimal consumption plan, the corresponding optimal portfolio is obtained in a standard way, as the hedging portfolio for the paying dividend rate $(e - c^f(\nu_x),\ \zeta^f(\nu_x))$ contingent claim. □

By integration with respect to ν of (3.2) and taking care to the integrability conditions, it follows that

Corollary 3.2 *If the Inada condition holds, the free value and the free dual value function are linked by*

$$\int_\nu^\infty (X_0^f(z) + I_0)dz = J(\nu) - \nu\, I_0 \tag{3.3}$$

If the Inada's condition does not holds, the same method leads to

$$\int_\nu^{\bar{\nu}} (X_0^f(z) + I_0)dz = J(\nu) - \nu\, I_0 - E(\int_0^T u(0,s)\, ds - g(0))$$

where $\bar{\nu} = \inf\{\nu, X_0^f(\nu) = -I_0\}$. The needed presence of the expectation operator in the right member is due to the eventually stochastic character of the utility functions.
The corollary leads to the well known result $J'(\nu) = -X_0^f(\nu)$.

3.2 The general case : a singular problem

As in the free case, we apply the Kuhn-Tucker multiplier method in order to linearize the budget constraint and introduce a stochastic control problem which is the dual of the constrained problem. The budget constraint under the liquidity constraint stated as $\inf_{D\in\mathcal{D}}(x - E(\int_0^T H_s D_s(c_s - e_s)ds + H_T D_T \zeta)) \leq 0$ involves a parameter D, therefore the dual problem is depending on this parameter. In this section, we solve this dual problem, establish the basic properties of the dual value function and prove the existence of an optimal control D^*.

Existence of an optimal control

The dual problem is a minimization problem, defined as

$$\Phi(\nu) = \inf_{D \in \mathcal{D}} J(\nu; D),$$

where

$$J(\nu; D) \stackrel{def}{=} \sup_{(c,\zeta)\in H_2^+ \times L_2^+} E\Big\{\int_0^T u(c_t,t)\,dt + g(\zeta) - \nu(\int_0^T H_s D_s(c_s - e_s)\,ds + H_T D_T^- \zeta)\Big\}$$

It follows from the properties of utility functions that

$$J(\nu; D) = E\Big\{\int_0^T [\tilde{u}(\nu H_t D_t, t) \;+ \nu H_t D_t e_t]\,dt + \tilde{g}(\nu H_T D_T^-)\Big\}. \tag{3.4}$$

The dual problem appears now as a singular control problem, with controlled dynamics $(H_t D_t; 0 \leq t \leq T)$ where $(D_t, t \geq 0)$ is a non-increasing process bounded by 1.

Remark: It would seem easier to restrict our attention to the processes D which are absolutely continuous on the form $dD_t = -\delta_t D_t dt$. In this case $H_s D_s = H_s^\delta$ where $dH_t^\delta = H_t^\delta[-(r_t + \delta_t)\,dt - \theta_t dW_t]$. This is the dynamics of a controlled diffusion where the control process δ is non-negative. However, there does not exist optimal control in this family (an optimal one would be obtained for a process δ which takes only two values 0 and $+\infty$.)

If $(r_t, t \geq 0)$ and $(\theta_t, t \geq 0)$ are deterministic, an analoguous problem, known as fuel problem, has been solved in Taksar (1985), El Karoui and Karatzas (1991), and other authors using the fact that the first derivative of the value function is related with an optimal stopping problem. We use here similar arguments. Let us denote by $Y_0(\nu)$ the initial value of the fair-value process associated with the free optimal consumption plan $(c^f(\nu), \zeta^f(\nu))$, i.e.,

$$Y_0(\nu) \stackrel{def}{=} \sup\nolimits_{\tau\in\mathcal{T}} E\Big\{\int_0^\tau H_s\{-\tilde{u}'(\nu H_s, s) - e_s\}\,ds - H_T \tilde{g}'(\nu H_T)\mathbb{1}_{\tau=T}\Big\}. \tag{3.5}$$

In the sequel, $Y_0(\nu)$ is simply called the **fair-value**. We also introduce the process

$$Y_t(\nu) \stackrel{def}{=} \sup\nolimits_{\tau\in\mathcal{T}(t)} E\Big\{\int_t^\tau H_s^t\{-\tilde{u}'(\nu H_s^t, s) - e_s\}\,ds - H_T^t \tilde{g}'(\nu H_T^t)\mathbb{1}_{\tau=T}\,|\mathcal{F}_t\Big\}. \tag{3.6}$$

Using this notation, the fair-value process associated with the free optimal consumption plan $(c^f(\nu), \zeta^f(\nu))$, defined as in (2.9) by

$$\sup\nolimits_{\tau\in\mathcal{T}(t)} E\Big\{\int_t^\tau H_s^t\{-\tilde{u}'(\nu H_s, s) - e_s\}\,ds - H_T^t \tilde{g}'(\nu H_T)\mathbb{1}_{\tau=T}\,|\mathcal{F}_t\Big\}. \tag{3.7}$$

is equal to $Y_t(\nu H_t)$ and is called the fair-value process.

Let us remark that the maps $\nu \to Y_0(\nu)$ and $\nu \to Y_t(\nu)$ are non-increasing, and that the optimal stopping time in (3.5) is defined as $\sigma(\nu) = \inf\{t, 0 \leq t < T \,|\, Y_t(\nu H_t) = 0\} \wedge T$. The main result of this section is the following

Theorem 3.3 *Let Y_0 the fair-value defined in* (3.5).
a. *Let Φ be the value fonction of the dual problem, and J be the value function for the free dual problem. Then*

$$\Phi(\nu) = J(\nu) - \int_0^{\nu} (Y_0(z) - X_0^f(z))\, dz\,. \tag{3.8}$$

b. *Furthermore, there exists an optimal control* [4] *for the dual problem, that is a process $D^*(\nu) \in \mathcal{D}$ such that $\Phi(\nu) = J(\nu; D^*(\nu))$.*

Remarks : a. The function $z \to Y_0(z) - X_0^f(z)$ is integrable in a neighbourhood of 0, this is no more the case for $X_0^f(z)$.
b. From the above theorem and Corollary 3.2, $-\Phi'(\nu) = Y_0(\nu) \geq 0$, an inequality which has a certain importance in He and Pagès (1993).

Proof: The proof of this main result follows from a change of variables formula. Instead of working directly with Φ, we study the difference between the free dual value function J and the dual value function Φ and we will rely this quantity with

$$Y_0(z) - X_0^f(z) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}} E\Big(\int_\tau^T H_s\,(\tilde{u}'(zH_s, s) + e_s)\, ds\; + \mathbb{1}_{\tau<T} H_T\, \tilde{g}'(zH_T)\Big)\,.$$

which, as we have seen in the comments following Theorem 2.1, gives the value of an American put on the negative part of the free wealth :

$$Y_0(z) - X_0^f(z) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}} E(\,[X_\tau^f]^-)$$

From the definition of the free value function and of $J(\nu, D)$, we have:

$$\begin{aligned} J(\nu) - J(\nu; D) \;\; &= \;\; E\Big\{\int_0^T (\tilde{u}(\nu H_s, s) + \nu H_s e_s)\, ds + \tilde{g}(\nu H_T) \\ &\qquad - \int_0^T (\tilde{u}(\nu H_s D_s, s) + \nu H_s D_s e_s)\, ds + \tilde{g}(\nu H_T D_{T_-})\Big\} \end{aligned}$$

Let us introduce the right-continuous, decreasing inverse $(\gamma_\theta, 0 \leq \theta \leq 1)$ of the decreasing process $(D_t, t \geq 0)$ to give an integral representation of quantities $\tilde{u}(\nu h, s) - \tilde{u}(\nu h D_s, s)$. The variable γ_θ is a stopping time and satisfies $\gamma_\theta = \inf\{s | D_s \leq \theta\}$. The set equality

$$\{D_s \leq \theta\} = \{\gamma_\theta \leq s\}$$

will be of constant use in what follows. We state without proof the obvious lemma which is the key for the main result

[4] In the sequel, we give an explicit construction of this optimal control.

Lemma 3.4 *We have*

$$\tilde{u}(\nu h, s) - \tilde{u}(\nu h D_s, s) = \int_{[0,1]} \nu h\, \tilde{u}'(\nu h y) 1\!\!1_{\{\gamma_y \le s\}}\, dy = \int_0^\nu h\, \tilde{u}'(hz) 1\!\!1_{\{\gamma_{\frac{z}{\nu}} \le s\}}\, dz\,.$$

Applying the lemma gives the inequality

$$J(\nu) - J(\nu; D) = E\Big\{\int_0^\nu dz \left(\int_0^T H_s\, 1\!\!1_{\{\gamma_{\frac{z}{\nu}} \le s)\}} (\tilde{u}'(zH_s, s) + e_s)\, ds\right) + H_T \tilde{g}'(zH_T) 1\!\!1_{\{\gamma_{\frac{z}{\nu}} < T\}}\Big\}$$

$$\le \int_0^\nu (Y_0(z) - X_0^f(z))\, dz$$

We minimize $J(\nu, D)$ over D and obtain

$$\Phi(\nu) \ge J(\nu) - \int_0^\nu (Y_0(z) - X_0^f(z))\, dz\,. \tag{3.9}$$

Conversely, let ν be given and introduce the family of optimal stopping times $(\sigma(z),\, z \le \nu)$ such that

$$Y_0(z) - X_0^f(z) = E\Big\{\int_{\sigma(z)}^T H_s(-\tilde{u}'(zH_s, s) - e_s)\, ds - H_T \tilde{g}'(zH_T) 1\!\!1_{\sigma(z) < T}\Big\}.$$

This identity implies that the equality holds in (3.9) after the identification of $\sigma(z)$ with a non-increasing process on the form $\gamma_{\frac{z}{\nu}}$, or in an equivalent way of $\sigma(\theta\nu)$ and γ_θ.

Since $(-\tilde{u}', -\tilde{g}')$ are non-increasing functions, the map $\nu \to \sigma(\nu)$ is non-increasing, and right continuous (see below for a rigourous proof and more complete properties). Let us denote by M the right-continuous inverse of σ,

$$\{s < \sigma(\nu)\} \;=\; \{\nu < M(s)\}\,. \tag{3.10}$$

The right inverse of the map $\theta \in [0, 1] \to \sigma(\theta\nu)$ is the right continuous process $D^*(\nu)$

$$D_s^*(\nu) = \left[\frac{1}{\nu} M(s)\right] \wedge 1\,.$$

Applying the lemma, we have

$$\int_0^\nu [Y_0(z) - X_0^f(z)]\, dz = J(\nu) - J(\nu; D^*(\nu))\,.$$

Therefore, the decreasing process $D^*(\nu) \in \mathcal{D}$ is optimal. The map $\nu \to D_t^*(\nu)$ is non-increasing and continuous. Furthermore, $D_s^*(\nu) = 1$ for $s < \sigma(\nu)$ and $D_s^*(\nu) < 1$ if $s > \sigma(\nu)$. The process $D^*(\nu)$ decreases only on the set $Y_s(\nu H_s) = 0$, i.e., $\int_0^T Y_s(\nu H_s)\, dD_s^*(\nu) = 0$. □

Remark : If $e \equiv 0$ and if the Inada condition prevails, $\sigma(\nu) = T$, $\forall \nu$, $M \equiv \infty$ and $D^* \equiv 1$. If $e \equiv 0$ and if the Inada condition does not holds, $\sigma(\nu) = T \wedge \inf\{s \,|\, \nu H_s \geq u'(0)\}$.

Furthermore, the optimal process $D^*(\nu)$ is the minimal one with the following meaning: let $\Delta(\nu)$ be an optimal process, i.e., such that $\Delta(\nu) \in \mathcal{D}$, $\Phi(\nu) = J(\nu, \Delta(\nu))$ and denote by γ its inverse. From right continuity of $\gamma(\cdot)$ and $\sigma(\cdot)$ and minimal character of σ, we deduce that $\sigma(\theta\nu) \leq \gamma(\theta)$ or equivalently $(D^*_t(\nu) \leq \Delta_t(\nu); t \geq 0)$. In particular, any optimal process $\Delta(\nu)$ is equal to 1 over $[0, \sigma(\nu)]$.

An expression for the fair value

We establish now properties of the process $(Y_t(\nu), t \geq 0)$ which will be important to prove that $(Y_t(\nu H_t D^*_t(\nu)), t \geq 0)$ is a fair price process, i.e., is a positive wealth which satisfies (2.3).

Proposition 3.5 *Let $D^*(\nu)$ be the optimal process for the dual problem defined in theorem 3.3. The following relations hold, where $H^*_t(\nu) = \nu H_t D^*_t(\nu)$ and $H^*_{T^-}(\nu) = \nu H_T D^*_{T^-}(\nu)$:*

$$Y_0(\nu) = E\Big(\int_0^T H_t D^*_t(\nu)\{-\tilde{u}'(H^*_t(\nu), t) - e_t\}\, dt - H_T D^*_{T^-}(\nu) \tilde{g}'(H^*_{T^-}(\nu)\,)\Big)\,, \quad (3.11)$$

$$Y_0(\nu) = E\Big(\int_0^T H_t\{-\tilde{u}'(H^*_t(\nu), t) - e_t\}\, dt - H_T\, \tilde{g}'(H^*_{T^-}(\nu)\,)\Big)\,. \quad (3.12)$$

In particular, the map $\nu \to Y_0(\nu)$ is non-increasing, continuous on $\mathbb{R}^*_+$ and strictly decreasing up to its first hitting time to $-I_0$; furthermore, $\lim_{\nu\to 0} Y_0(\nu) = \infty$ and $\lim_{\nu\to\infty} Y_0(\nu) = -I_0$.

Proof: The stopping time $\sigma(z)$ is an optimal stopping time for the problem associated with $(-\tilde{u}'(zH_t, t), t \geq 0, -\tilde{g}'(zH_T))$. Therefore

$$E\Big(\int_0^{\sigma(\lambda z)} H_t(-\tilde{u}'(zH_t, t) - e_t)\, dt - H_T \tilde{g}'(zH_T) 1\!\!1_{\{\sigma(\lambda z) = T\}}\Big)$$

$$\leq E\Big(\int_0^{\sigma(z)} H_t(-\tilde{u}'(zH_t, t) - e_t)\, dt - H_T \tilde{g}'(zH_T) 1\!\!1_{\{\sigma(z) = T\}}\Big) = Y_0(z) < \infty.$$

The non-increasing property of the map $-\tilde{u}'$ implies the non-increasing property of the map $z \to \sigma(z)$ which allows us to deduce that, for $\lambda > 1$,

$$E\Big(\int_{\sigma(\lambda z)}^{\sigma(z)} H_t(-\tilde{u}'(zH_t, t) - e_t)\, dt - H_T \tilde{g}'(zH_T) 1\!\!1_{\{\sigma(z) = T > \sigma(\lambda z)\}}\Big) \geq 0\,.$$

We would like to integrate this inequality with respect to z on the interval $[0, \nu]$. However, the functions $-\tilde{u}'$ and $-\tilde{g}'$ are too large in a neighbourough of zero, and we need to integrate with respect to z only on the interval $[\epsilon,\nu]$ with $\epsilon>0$ to obtain

$$E\Big(\int_\epsilon^\nu dz\left[\int_{\sigma(\lambda z)}^{\sigma(z)} H_t(-\tilde u'(zH_t,t)-e_t)\,dt - H_T\tilde g'(zH_T)\mathbb{1}_{\{\sigma(z)=T>\sigma(\lambda z)\}}\right]\Big)\geq 0\,,$$

which, from the non-increasing property of $-\tilde u'$ and $-\tilde g'$ and the link between σ and M expressed in (3.10) leads to

$$E\Big(\int_\epsilon^\nu dz\left[\int_0^{\sigma(\epsilon)} H_t\,\mathbb{1}_{\{\frac{1}{\lambda}M_t\leq z<M_t\}}[-\tilde u'((\frac{1}{\lambda}M_t\vee\epsilon)H_t,t)-e_t]\,dt\right.$$

$$\left.-H_T\,\tilde g'((\frac{1}{\lambda}M_T^-\vee\epsilon)H_T)\mathbb{1}_{\{\frac{1}{\lambda}M_T^-\leq z<M_T^-\}}\right]\Big)\geq 0\,.$$

Applying Fubini's theorem gives

$$E\Big(\int_0^{\sigma(\epsilon)} H_t\left[-\tilde u'\left((\frac{M_t}{\lambda}\vee\epsilon)H_t,t\right)-e_t\right]\left[\int_\epsilon^\nu \mathbb{1}_{\{\frac{1}{\lambda}M_t\leq z<M_t\}}dz\right]\,dt$$

$$-H_T\,\tilde g'(\,(\frac{M_T^-}{\lambda}\vee\epsilon)H_T)\,\left[(\epsilon\vee[M_T^-\wedge\nu])-([\frac{M_T^-}{\lambda}\vee\epsilon]\wedge\nu)\right]\Big)\geq 0\,. \tag{3.13}$$

On the set $\{\frac{M_t}{\lambda}<\nu\}\cap\{t<\sigma(\epsilon)\}$, the quantity $\int_\epsilon^\nu \mathbb{1}_{\{\frac{1}{\lambda}M_t\leq z<M_t\}}dz$ is equal to $(M_t\wedge\nu)-(\frac{M_t}{\lambda}\vee\epsilon)$, furthermore the inequalities

$$0\leq\frac{\lambda}{\lambda-1}\left(\int_\epsilon^\nu \mathbb{1}_{\{\frac{1}{\lambda}M_t\leq z<M_t\}}dz\right)\leq\nu\mathbb{1}_{\{\frac{1}{\lambda}M_t<\nu\}}$$

and $-\tilde u'((\frac{1}{\lambda}M_t\vee\epsilon)H_t,t)\leq-\tilde u'(\epsilon H_t,t)$ hold. Let us multiply the left member of (3.13) by $\frac{\lambda}{\lambda-1}$ and let λ go to 1; the right-continuity of $\sigma(.)$, the boundness properties and the convergence towards $M_t\mathbb{1}_{[\sigma(\nu),T]}$ of $\dfrac{\lambda(M_t\wedge\nu)-M_t}{\lambda-1}\mathbb{1}_{[\sigma(\lambda\nu),T]}$ when λ goes to 1 allow us to pass to the limit. We obtain

$$E\Big(\int_{\sigma(\nu)}^{\sigma(\epsilon)} H_tM_t[-\tilde u'(M_tH_t,t)-e_t]\,dt-H_T(\epsilon\vee M_T^-)\,\tilde g'(H_T(M_T^-\vee\epsilon)\,)\Big)\geq 0\,.$$

We split the integral in two parts

$$E\Big(\int_{\sigma(\nu)}^{\sigma(\epsilon)} H_tM_t\,(-\tilde u'(M_tH_t,t))\,dt\Big)\quad\text{and}\quad E\Big(\int_{\sigma(\nu)}^{\sigma(\epsilon)} H_tM_te_t\,dt\Big)\,.$$

It remains to let ϵ go to zero and remark that $\sigma(\epsilon)$ converges towards T. The integral $E(\int_{\sigma(\nu)}^T H_tM_te_t\,dt)$ converges, since, on $\{\sigma(\nu)\leq t\}$, the process M is bounded by ν. Using the non-negativity of $-\tilde u'$, it follows

$$E\Big(\int_{\sigma(\nu)}^{T} H_tM_t\{-\tilde u'(M_tH_t,t)-e_t\}\,dt-H_TM_T^-\tilde g'(H_TM_T^-\,)\Big)\geq 0\,.$$

The inequality

$$E\Big(\int_{\sigma(\nu)}^{T} H_t D_t^*(\nu)\{-\tilde{u}'(\nu H_t D_t^*(\nu),t)-e_t\}\,dt - H_T D_{T-}^*(\nu)\tilde{g}'(\nu H_T D_{T-}^*(\nu))\Big) \geq 0$$

follows from the equality $\nu D_t^*(\nu) = M_t$ on the set $\{\sigma(\nu)<t\}=\{M_t<\nu\}$.

If $\lambda<1$, the same arguments lead to an inequality similar to (3.13) where the lower bound of the integral does not depend on λ :

$$E\Big(\int_{\sigma(\nu)}^{T} H_t\,(-\tilde{u}'(\nu \wedge \frac{M_t H_t}{\lambda},t)-e_t)\Big(\int_{\epsilon}^{\nu} 1\!\!1_{M_t\leq z<\frac{1}{\lambda}M_t} dz\Big)\,dt$$

$$-H_T\,[(\frac{M_{T-}}{\lambda}\wedge\nu)-M_T^-]\,\tilde{g}'(\frac{H_T M_T^-}{\lambda})\Big) \leq 0\,.$$

By multipling the left member by $\frac{\lambda}{1-\lambda}$ we obtain, letting λ go to 1 and ϵ to zero

$$E\Big(\int_{\sigma(\nu)}^{T} H_t D_t^*(\nu)\{-\tilde{u}'(\nu H_t D_t^*(\nu),t)-e_t\}\,dt - H_T D_{T-}^*(\nu)\tilde{g}'(\nu H_T D_{T-}^*(\nu))\Big) \leq 0\,.$$

The first equality in the theorem follows.

The second equality in the theorem is obtained in the same manner, using the family of stopping times $\sigma(z+\lambda)$ instead of $\sigma(\lambda z)$ which leads, for $\lambda>0$ to the inequality

$$E\Big[\int_{\sigma(\nu+\lambda)}^{\sigma(\epsilon)} H_t(\,[(M_t\wedge\nu)\vee\epsilon]-[(\,(M_t-\lambda)\vee\epsilon)\wedge\nu])\,(-\tilde{u}'(H_t[(M_t-\lambda)\vee\epsilon],t)-e_t)\,dt$$

$$-H_T\;([(M_T^-\wedge\nu)\vee\epsilon]-[(\,(M_T^- -\lambda)\vee\epsilon)\wedge\nu])\,\tilde{g}'([(M_T^- -\lambda)\vee\epsilon])\Big] \geq 0\,,$$

and for $\lambda<0$

$$E\Big(\int_{\sigma(\nu)}^{\sigma(\epsilon)} H_t([(M_t+\lambda)\wedge\nu]-M_t)\,(-\tilde{u}'(H_t(M_t+\lambda),t)-e_t)\,dt$$

$$-H_T([(M_{T_-}+\lambda)\wedge\nu]-M_{T_-})\,\tilde{g}'((M_{T-}+\lambda))\Big) \leq 0\,.$$

Dividing by λ and letting λ go to zero give the result.

We study the map $\nu\rightarrow Y_0(\nu)$. From (3.12), it can be proved that this map is continuous, since $\nu D_t^*(\nu) = M_t\wedge\nu$. When ν goes to zero, using (3.12) and the convergence of $\nu D_t^*(\nu)$ towards zero, $Y_0(\nu)$ goes to ∞. When ν goes to ∞, using (3.11) and the equality $\nu D_t^*(\nu) = M_t\wedge\nu$, we write (in the case $g=0$ for simplicity)

$$Y_0(\nu) = E\left(\int_0^{\sigma(\nu)} H_t[-\tilde{u}'(\nu H_t,t)]\,dt\right) - E\left(\int_0^{\sigma(\nu)} H_t e_t dt\right)$$

The quantity $E(\int_0^{\sigma(\nu)} H_t[-\tilde{u}'(\nu H_t,t)]\,dt)$ is bounded by $E(\int_0^T H_t[-\tilde{u}'(\nu H_t,t)]\,dt)$ and converges to zero, whereas the quantity $-E(\int_0^{\sigma(\nu)} H_t e_t dt\ )$ converges to $-I_0$, due to the convergence of $\sigma(\nu)$ towards T.

□

3.3 Boundary and optimal stopping time

Let us recall that the optimal stopping time $\sigma(\nu)$ such that

$$Y_0(\nu) = E\Big\{\int_0^{\sigma(\nu)} H_s\{-\tilde{u}'(\nu H_s,s) - e_s\}\,ds - H_T\tilde{g}'(\nu H_T)1\!\!1_{\sigma(\nu)=T}\Big\}$$

is defined as the first time before T where the process $(Y_t(\nu H_t), t \geq 0)$ hits zero (if this time exists or T otherwise).

Proposition 3.6 *Define the stochastic boundary, for $t < T$*

$$b(t) = \inf\{\nu\,;\, Y_t(\nu) = 0\}\,.$$

Then

$$\sigma(\nu) = T \wedge \inf\{t < T\,;\, Y_t(\nu H_t) = 0\} = T \wedge \inf\{t < T\,;\, \nu H_t \geq b(t)\}$$

and the inverse of the right-continuous map $\nu \to \sigma(\nu)$ satisfies

$$M_t = \inf_{s\leq t} \frac{b(s)}{H_s}$$

PROOF:

Let us study the properties of $\sigma(z)$. The function $z \to \sigma(z)$ is non-increasing: this follows from the non-increasing property of $z \to Y_0(z)$ and the caracterisation of $\sigma(z)$ as $\sigma(z) = \inf\{t|Y_t(zH_t) = 0\}$. In order to establish the right-continuity of σ, let z_n be a sequence such that $z_n > z$ and $z_n \to z$. Then, denoting by σ^* the limit of $\sigma(z_n)$, the non-increasing property leads to $\sigma^* \leq \sigma(z)$. For any stopping time τ, the optimal character of $\sigma(z_n)$ and the non-increasing property of $-\tilde{u}'$ give the following inequalities

$$\begin{aligned} E\left(\int_0^{\tau} H_t(-\tilde{u}'(z_nH_t,t) - e_t)\,dt\right) &\leq E\left(\int_0^{\sigma(z_n)} H_t(-\tilde{u}'(z_nH_t,t) - e_t)\,dt\right) \\ &\leq E\left(\int_0^{\sigma(z_n)} H_t(-\tilde{u}'(zH_t,t) - e_t)\,dt\right) \end{aligned}$$

and, letting n go to infinity,

$$E\left(\int_0^{\tau} H_t(-\tilde{u}'(zH_t,t) - e_t)\,dt\right) \leq E\left(\int_0^{\sigma^*} H_t(-\tilde{u}'(zH_t,t) - e_t)\,dt\right)$$

The stopping time σ^* is therefore optimal, and from the minimal character of $\sigma(z)$ the inequality $\sigma^* \geq \sigma(z)$ follows.

Since $-\tilde{u}'$ and $-\tilde{g}'$ are non-negative functions which are non-increasing with respect to the first argument, the process $Y(\nu)$ is non-negative and non-increasing with respect to ν, therefore $\{\nu\,; Y_t(\nu) = 0\}$ is an interval of the form $(b(t), +\infty]$ where $(b(t), t \geq 0)$ is an adapted process. Then, the equality $\inf\{t < T\,;\, Y_t(\nu H_t) = 0\} = \inf\{t\,, t < T \,|\, \nu H_t \geq b(t)\}$ follows. Let us remark that $b(t) > 0$, since, for each t, $Y_t(\nu)$ converges to infinity when ν converges to zero.
It is obvious that, for any ν, the inequality $G_t \stackrel{def}{=} \inf_{s\leq t} \dfrac{b(s)}{H_s} \geq \nu$ holds on the set $\{\sigma(\nu) > t\}$, therefore $G_t \geq M_t$. In the other hand, consider that, on the set $G_t > \nu$, the inequality $\sigma(\nu) > t$ holds which implies that $M_t \geq G_t$, and the equality is now obvious for $t < T$ □

Reflection to the boundary

The equality $M_t = \inf_{s\leq t} \dfrac{b(s)}{H_s}$ establishes that $\nu D_t^*(\nu) \leq \dfrac{b(t)}{H_t}$, therefore the process $H^*(\nu)$ remains below the boundary, i.e., $H_t^*(\nu) \leq b(t)$, $\forall t \leq T$. The equality

$$\frac{d(H_t D_t^*)}{H_t D_t^*} = \frac{dD_t^*}{D_t^*} - (r_t dt + \theta_t dW_t)$$

can be interpreted as a reflection problem . The process $D^*(\nu)$ acts as a local time for the reflection along the stochastic boundary $b(t)$. The process $D^*(\nu)$ is the unique one to have this property.
This reflection is more obvious if we study the logarithm $\xi_t = \ln H_t D_t^*$ which is $\xi_t = \ln H_t + D_t^*$. If $\nu H_t \geq b(t)$, the optimal strategy is to throw the process at the boundary $b(t)$. □

3.4 The primal problem

¿From now on, we introduce, for simplicity, the notation

$$c_t^*(\nu) = -\tilde{u}'(H_t^*(\nu), t), \quad \zeta^*(\nu) = -\tilde{g}'(H_{T^-}^*(\nu))$$

and recall that

$$\begin{aligned} c_t^*(\nu) &= \operatorname{Arg\,max}\{u(c,t) - cH_t^*(\nu)\} && (3.14)\\ \zeta^*(\nu) &= \operatorname{Arg\,max}\{g(\zeta) - \zeta\, H_{T^-}^*(\nu)\} && (3.15)\end{aligned}$$

We have proved that, for x large enough, we can fit ν such that $Y_0(\nu_x) = x$. Using the concise notation $D_t^{*x} = D_t^*(\nu_x)$ and similar notation for ζ^{*x} and c^{*x}, we will establish that (c^{*x}, ζ^{*x}) is optimal for the primal problem over the all superstrategies and prove that this consumption plan is feasible, i.e., associated with a self-financing strategy and a non-negative current wealth. Using different approach, He and Pagès (1993) and Cuoco (1996) obtained similar results. Let us emphasize that, from the two representations (3.11, 3.12), we are able to deduce the major fact that the optimality is achieved in the class of x-feasible strategies, which is not done in the above papers.

Optimal consumption plan

Theorem 3.7 *Let H^* be the optimal solution for the dual problem. For a given endowment $x > -I_0$, there exists a multiplicator ν_x such that the consumption plan*

$$c^{*x} = -\tilde{u}'(H^{*x}_{\cdot}, \cdot) \quad \zeta^{*x} = -\tilde{g}'(H^{*x}_{T-})$$

is an optimal feasible strategy: the associated wealth

$$X^*_t = E\Big(\int_t^T H^t_s(c^*_s - e_s)\, ds + H^t_T \zeta^* | \mathcal{F}_t\Big)\,.$$

remains non-negative. The value function of the primal problem is $\Phi(\nu_x)$.

Proof: The definition of c^{*x} and ζ^{*x} implies that for each (c, ζ)

$$\begin{aligned} E\Big(\int_0^T u(c_t, t)\, dt + g(\zeta)\Big) \; &\leq E\Big(\int_0^T u(c^{*x}_t, t)\, dt + g(\zeta^{*x})\Big) \\ &-\nu_x\Big\{x - E\Big(\int_0^T H_t D^{*x}_t (c_t - e_t)\, dt + H_T D^{*x}_{T-}\zeta\Big)\Big\}. \end{aligned}$$

Therefore the consumption plan (c^{*x}, ζ^{*x}) is optimal among all the consumption plans (c, ζ) which satisfy

$$E\Big(\int_0^T H_t D^{*x}_t (c_t - e_t)\, dt + H_T D^{*x}_{T-}\zeta\Big) \leq x \tag{3.16}$$

¿From Theorem 2.1.c, it follows that any x-affordable consumption plan satisfies (3.16). At this stage, the consumption plan (c^{*x}, ζ^{*x}) is optimal in the class of superhedging stategies financed from an initial wealth equal to x. However, using similar arguments than in proposition 3.5, it can be establish that $Y_t(\nu)$ satisfies

$$\begin{aligned} Y_t(\nu) \;&=\; E\Big(\int_t^T H^t_s D^{t*}_s(\nu)\{-\tilde{u}'(H^{t*}_s(\nu), s) - e_s\}\, ds - H^t_T D^{t*}_{T-}(\nu)\tilde{g}'(H^{t*}_{T-}(\nu)\,)|\mathcal{F}_t\Big) \\ &=\; E\Big(\int_t^T H^t_s\{-\tilde{u}'(H^{t*}_s(\nu), s) - e_s\}\, ds - H^t_T\tilde{g}'(H^{t*}_{T-}(\nu)\,)|\mathcal{F}_t\Big) \end{aligned} \tag{3.17}$$

where $\nu H^t_s D^{t*}_s(\nu) = H^{t*}_s(\nu)$, and $D^*_t\, D^{t*}_s = D^*_s$.

The equality (3.17) provides that the process $X_t^{*x} \overset{def}{=} Y_t(H_t^{*x})$ is not only the fair-value of the optimal consumption plan, but also the free price of this consumption plan since

$$X_t^{*x} = E\Big(\int_t^T H_s^t(c_s^{*x} - e_s)\, ds + H_T^t \zeta^{*x} | \mathcal{F}_t\Big)$$

and now obviously $X_t^* = Y_t(H_t^{*x}) \geq 0, \forall t$. In particular, there exists a portfolio π^* such that

$$\begin{cases} dX_t^{*x} &= r_t X_t^{*x} dt - (c_t^{*x} - e_t)dt + \pi_t^*(dW_t + \theta_t dt), \\ X_0^{*x} &= x, \quad X_T^{*x} = \zeta^{*x} \end{cases}$$

This portfolio is given via the representation theorem. □

Remark: The optimal pair does not satisfy the square integrability conditions. Nevertheless $E\Big(\int_0^T H_t c_t^{*x} dt\Big) < \infty$ and $E(H_T \zeta^{*x}) < \infty$.

Properties of the optimal consumption plan

Let us describe more precisely the links between the optimal wealth and the optimal consumption.

Theorem 3.8 *The optimal consumption plan*

$$c^{*x} = -\tilde{u}'(H^{*x}, \cdot) \quad \zeta^{*x} = -\tilde{g}'(H_T^{*x})$$

can be expressed in a feedback form as

$$c_t^{*x} = -\tilde{u}'(\mathcal{Y}(t, X_t^{*x}), t), \quad \zeta^{*x} = -\tilde{g}'(\mathcal{Y}(T, X_T^{*x})$$

where $\mathcal{Y}(t, \cdot)$ *is the inverse of* $Y_t(\cdot)$. *Furthermore,* $0 \leq c_t^{*x} \leq e_t$ *on the set* $X_t^{*x} = 0$.

PROOF: From the two equalities $X_t^{*x} = Y_t(H_t^{*x})$ and $c_t^{*x} = -\tilde{u}'(H_t^{*x}, t)$ we can deduce a feedback representation of the optimal consumption, as it is well known in the no-income case. We have studied the decreasing properties of $Y_0(\cdot)$ and established the decreasing property up to the first hitting time of zero, i.e., on the interval $]0, b(0)[$. From

$$Y_t(\nu) = E\Big(\int_t^T H_s^t\{-\tilde{u}'(H_s^{t*}(\nu), s) - e_s\}\, ds - H_T^t \tilde{g}'(H_{T^-}^{t*}(\nu)\,)|\mathcal{F}_t\Big)$$

we deduce that $Y_t(\cdot)$ is strictly decreasing on $]0, b(t)[$ and, since $H_t^*(\nu)$ remains below the boundary, the map $\nu \to Y_t(\nu H_t^*(\nu))$ is strictly decreasing and admits an inverse $\mathcal{Y}(t, \cdot)$, which is adapted, such that $H_t^*(\nu_x) = \mathcal{Y}(t, X_t^*)$. The optimal consumption can be expressed in terms of the optimal wealth $c_t^* = -\tilde{u}'(\mathcal{Y}(t, X_t^*), t)$. As a check, remark that, for a given level of the wealth, this optimal consumption

is smaller than the optimal consumption in the free problem. This representation is similar to the free-case, the only difference is the choice of the process X^* instead of the free wealth X^f. If the wealth is equal to zero (at the boundary), the agent can have a non zero consumption.
Let us be more precise : from Tanaka's formula, and using that X^* is non-negative

$$dX_t^* = \mathbb{1}_{X_t^* \neq 0}\, dX_t^* - \mathbb{1}_{X_t^*=0}\, dX_t^* + \mathbb{1}_{X_t^*=0}\, dL_t^*$$

where L^* is the increasing local time process. An identification with $\mathbb{1}_{X_t^* \neq 0}\, dX_t^* + \mathbb{1}_{X_t^*=0}\, dX_t^*$ gives $\mathbb{1}_{X_t^*=0} dL_t^* = 2\mathbb{1}_{X_t^*=0} dX_t^*$. The equality of the bounded variation parts of the two decompositions of X^* leads to $-(c_t^* - e_t)\mathbb{1}_{X_t^*=0}\, dt = \mathbb{1}_{X_t^*=0}\, dL_t^*$, and $c_t^* \leq e_t$ on $X_t^* = 0$. □

4 A classical Markovian situation

4.1 The general Markovian case

If the state price density H and the income process e are Markov processes with $de_t = \mu(t, e_t)dt + \sigma(t, e_t)dW_t$, the analogy with the results obtained in the no-income case can be drawn further, as it was done in He and Pagès (1993). For notational simplicity, we restrict our attention to the case

$$dH_t = -H_t[rdt + \theta dW_t], \quad de_t = e_t[\mu dt + \sigma dW_t]$$

where r, θ, μ and σ are constant coefficients. The free dual value function J is defined in a dynamic form as

$$J(t, \nu, \varepsilon) \stackrel{def}{=} E(\int_t^T (\tilde{u}(\nu H_s^t, s) + \nu H_s^t e_s)ds + \tilde{g}(\nu H_T^t)|e_t = \varepsilon)$$

and satisfies

$$\tilde{u}(\nu, t) + \nu\varepsilon + \mathcal{L}(J) = 0 \tag{4.1}$$

where

$$\mathcal{L}(J) = J_t - \nu r J_\nu + \varepsilon\mu J_\varepsilon + \frac{1}{2}\nu^2\theta^2 J_{\nu,\nu} + \frac{1}{2}\varepsilon^2\sigma^2 J_{\varepsilon,\varepsilon} - \nu\theta\varepsilon\sigma J_{\nu,\varepsilon}$$

and the terminal condition $J(T, \nu, \varepsilon) = \tilde{g}(\nu)$.
We define in a dynamic way, the dual value function $\Phi(t, \nu, e_t)$ where

$$\Phi(t, \nu, \varepsilon) \stackrel{def}{=} \min_{D \in \mathcal{D}} E(\int_t^T (\tilde{u}(\nu H_s^t D_s, s) + \nu H_s^t D_s e_s)ds + \tilde{g}(\nu H_T^t D_T^-)|e_t = \varepsilon)\,.$$

Restricting our attention to absolutely continous parameters D, our problem can be caracterized by mean of a variational inequality (cf. He and Pagès)

$$\begin{aligned}
\Phi_\nu &\leq 0 \\
\tilde{u}(\nu, t) + \nu\varepsilon + \mathcal{L}(\Phi) &\leq 0 \\
(\Phi_\nu)(\tilde{u}(\nu, t) + \nu\varepsilon + \mathcal{L}(\Phi)) &= 0 \\
\Phi(T, \nu, \epsilon) &= \tilde{g}(\nu)
\end{aligned}$$

which we write on the form

$$\max(\Phi_\nu, \tilde{u}(\nu,t) + \nu\varepsilon + \mathcal{L}(\Phi)) = 0\,.$$

The previous boundary $b(t)$ is related to the manifold that separates $\{(t,\nu,\epsilon)\,;\,\Phi_\nu(t,\nu,\epsilon) = 0\}$ and the continuation region $\{(t,\nu,\epsilon)\,;\,\Phi_\nu(t,\nu,\epsilon) < 0\}$. (Recall that $\Phi_\nu(t,\nu H_t, e_t) = -Y_t(\nu H_t)$.)

¿From (4.1) we can deduce a variational inequality for $\Psi(t,\nu,\varepsilon) = \Phi_\nu(t,\nu,\varepsilon)$. However, let us show how we can obtain this inequality on a direct way. The variational equation associated with the American put can be deduced from

$$\Psi(t,\nu,\varepsilon) = \sup_{\tau\geq t} E\left(\int_t^\tau H_s^t(-\tilde{u}'(\nu H_s^t, s) - e_s)\,ds - H_\tau^t \mathbb{1}_{\tau=T}\tilde{g}'(\nu H_T^t)|e_t = \varepsilon\right)\,.$$

Under the risk-neutral probability, the process $\hat{W}_t = W_t + \theta t$ is a Brownian motion

$$de_t = e_t[(\mu - \sigma\theta)dt + \sigma d\hat{W}_t]$$

$$dH_t = H_t[(-r+\theta^2)dt - \theta d\hat{W}_t]$$

and

$$\Psi(t,\nu,\varepsilon) = \sup_{\tau\geq t} E_Q\left(\int_t^\tau (-\tilde{u}'(\nu H_s^t, s) - e_s)\,ds - \mathbb{1}_{\tau=T}\tilde{g}'(\nu H_{T^t})|e_t = \varepsilon\right)\,.$$

is a solution of

$$\max(\Lambda\Psi(t,\nu), -\Psi(t,\nu)) = 0$$

where

$$\Lambda\Phi = \Phi_t + (\theta^2 - r)\Phi_\nu + \varepsilon(\mu - \theta\sigma)\Psi_\varepsilon + \frac{1}{2}\nu^2\theta^2\Psi_{\nu,\nu} + \frac{1}{2}\sigma^2(t,\varepsilon)\Psi_{\varepsilon,\varepsilon} - \varepsilon\nu\theta\sigma\Psi_{\nu,\varepsilon} - r\Psi - \tilde{u}' - \varepsilon$$

The optimal wealth is $X_t^* = \mathcal{X}(t, e_t, H_t^{*x})$ where

$$\mathcal{X}(t,\varepsilon,\nu) = E\left(\int_t^T H_s^t(-\tilde{u}'(H_s^{t*x}(\nu), s) - e_s)\,ds - H_T^t\tilde{g}'(H_{T_-}^{t*x}(\nu))|e_t = \varepsilon\right)\,.$$

Using Itô's formula, we obtain the optimal portfolio π^* by means of $\mathcal{X}$

$$[-H_t^{*x}\theta\frac{\partial\mathcal{X}}{\partial\nu} + \varepsilon\sigma\frac{\partial\mathcal{X}}{\partial\varepsilon}](t,\varepsilon,H_t^{*x}) = \pi_t$$

As in He and Pagès (1993) or Karatzas (1996), various formulae can be obtained. It sufficies to change H into H^*.

4.2 An example

We give a closed formula for the optimal consumption plan in the particular case where

- the dynamics of the state prices is given by

$$dH_t = -H_t(rdt + \theta dW_t)\,,$$

- the income flow is a geometrical Brownian motion

$$de_t = -e_t(\mu_e dt + \sigma_e dW_t), \qquad e_0 = \varepsilon.$$

- the coefficients r, θ, μ_e and σ_e are constant.
- the agent's preferences over consumption profiles are given by an HARA utility function $u(c,t)$ with convex conjugate function $\tilde{u}(\nu, t)$

$$\begin{cases} u(c,t) &=& e^{-\beta t}\frac{c^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1, \\ u(c,t) &=& e^{-\beta t} Ln(c) & \text{if } \gamma = 1, \\ -\tilde{u}'(\nu,t) &=& (\nu e^{\beta t})^{-\frac{1}{\gamma}}. \end{cases}$$

Let us introduce two conditions on the coefficients: under the first one, the income present value is finite, the second one implies that an optimal policy exists for the free case.
A1) The certainty equivalent present value I_0 of the lifetime labor income $I_0(\varepsilon) = E\int_0^{+\infty} H_t e_t\, dt$ is finite[5], i.e., $r + \mu_e - \theta\sigma_e > 0$. In this case $I_0(\varepsilon) = B\varepsilon < +\infty$, where $B > 0$ is given by $B^{-1} \stackrel{def}{=} r + \mu_e - \theta\sigma_e$.
A2) The free dual problem is well-posed, that is $-I_0(\varepsilon) < X_0^f(\nu) < \infty$. This is equivalent to

$$0 < X_0^f(\nu) + I_0(\varepsilon) = E\Big(\int_0^\infty H_t\,(\nu e^{\beta t} H_t)^{-\frac{1}{\gamma}}\, dt\Big) < \infty$$

is non-negative and finite, or to $0 < A < \infty$ where

$$A^{-1} \stackrel{def}{=} \frac{\beta}{\gamma} + (\frac{\gamma-1}{\gamma})(r + \frac{\theta^2}{2\gamma}).$$

[5] Let us recall that from Ito's formula

$$\begin{aligned} \frac{dH_t^a e_t^b e^{ct}}{H_t^a e_t^b e^{ct}} &= [c - ar - b\mu_e + \frac{1}{2}a(a-1)\theta^2 + \frac{1}{2}b(b-1)\sigma_e^2 + ab\theta\sigma_e]dt \\ &- (a\theta + b\sigma_e)dW_t. \end{aligned} \tag{4.2}$$

From this equality, it follows that

$$E(H_t^a e_t^b e^{ct}) = \varepsilon^b \exp([c - ar - b\mu_e + \frac{1}{2}a(a-1)\theta^2 + \frac{1}{2}b(b-1)\sigma_e^2 + ab\theta\sigma_e]t)$$

The free case

From Markovian properties, we deduce that at time t, the optimal free wealth is given by

$$X_t^f(\nu) = A(\nu e^{\beta t} H_t)^{-\frac{1}{\gamma}} - Be_t. \tag{4.3}$$

To force the initial wealth to be equal to x, the multiplier ν must satisfy

$$A\nu^{-\frac{1}{\gamma}} = x + B\varepsilon.$$

In the free case, as it was shown by Svensson and Werner (1991), Merton (1991), He and Pagès (1993) and Koo (1995) the optimal consumption profile at time 0 given by

$$c_0(\nu) = \nu^{-\frac{1}{\gamma}} = \frac{x + B\varepsilon}{A}$$

is a fraction of the wealth plus the income present value. This relation also holds at time t on the form $c_t(\nu) = \dfrac{X_t^f(\nu) + Be_t}{A}$.
From the equation (4.3), we deduce the dynamics of the optimal wealth

$$\begin{aligned} dX_t^f(\nu) &= -Bde_t + Ad(\nu H_t e^{\beta t})^{-\frac{1}{\gamma}} \\ &= Be_t(\mu_e dt + \sigma_e dW_t) + (X_t^f(\nu) + Be_t)\left[[\tfrac{1}{\gamma}(r-\beta) + \tfrac{1}{2\gamma}(\tfrac{\gamma+1}{\gamma})\theta^2]dt + \tfrac{\theta}{\gamma}dW_t\right] \end{aligned}$$

The optimal portfolio is $\pi_t = \dfrac{\theta}{\gamma}[X_t^f(\nu)] + (\sigma_e + \dfrac{\theta}{\gamma})Be_t$.

The constrained case

We have seen that the fair value $Y_0(\nu)$ is easily deduced from the price of an American option written on the negative part of the free value, that is,

$$Y_0(\nu) = X_0^f(\nu) - V_0(\nu),$$

where

$$V_0(\nu) \overset{def}{=} \sup_{\tau \in \mathcal{T}} E\left(H_\tau \left[A(\nu e^{\beta\tau} H_\tau)^{-\frac{1}{\gamma}} - Be_\tau\right]^-\right).$$

The associated one-dimensional stopping time problem

The function V_0 is the value function of an optimal stopping problem associated with a two dimensional Markov diffusion process $(\nu e^{\beta t} H_t, e_t;\ t \geq 0)$. From the homegeneity property of the pay-off, it is easy to transform this problem into an optimal stopping problem for a one-dimensional Markov process

$$\frac{V_0(\nu)}{\varepsilon} = \sup_{\tau \in \mathcal{T}} E\left[\frac{H_\tau e_\tau}{\varepsilon}(AZ_\tau - B)^-\right], \quad \text{where} \quad Z_t = \frac{(\nu e^{\beta t} H_t)^{-\frac{1}{\gamma}}}{e_t} \tag{4.4}$$

To make the transformation complete, let us introduce a change of probability measure and define a new probability measure Q^e, such that the Radon-Nikodym derivative of Q^e w.r.to P is the exponential martingale generated by $-(\theta+\sigma_e)W$.

Proposition 4.1 *Let $\widehat{V}(z)$ be defined as*

$$\widehat{V}(z) \stackrel{def}{=} \frac{V_0(\nu)}{\varepsilon} = \sup_{\tau\in\mathcal{T}} E_{Q^e}[e^{-\frac{\tau}{B}}(AZ_\tau - B)^-]\,. \tag{4.5}$$

The function $\widehat{V}$ is the value function of an optimal stopping time problem with respect to $(AZ-b)^-$ where the Markov process $(Z_t, 0\le t)$ is a geometrical Brownian motion under the probability measure Q^e with characteristics given by

$$\left\{\begin{array}{lcl} dZ_t &=& Z_t(\mu_Z dt + \sigma_Z dW_t^Z), \\ Z_0 &=& z = \nu^{-\frac{1}{\gamma}}\varepsilon^{-1}\,, \\ \mu_Z &=& B^{-1} - A^{-1}\,, \\ \sigma_Z^2 &=& \dfrac{\theta^2}{\gamma^2} + \sigma_e^2 + \dfrac{\theta\sigma_e}{\gamma}\,. \end{array}\right. \tag{4.6}$$

Proof : From (4.4)

$$dZ_t = Z_t\left[[\frac{r-\beta}{\gamma} + \mu_e + \frac{1+\gamma}{2\gamma^2}\theta^2 + \sigma_e^2 + \frac{\theta\sigma_e}{\gamma}]dt + (\frac{\theta}{\gamma}+\sigma_e)\,dW_t\right]\,.$$

Let us recall that $(H_t e_t, 0\le t)$ is a geometrical Brownian motion with discount rate $-B^{-1}$ and Gaussian part $-(\theta+\sigma_e)W$.
With respect to Q^e, W is a Brownian motion $\widehat{W}$ with a drift given by $-(\theta+\sigma_e)$. Therefore,

$$(\frac{\theta}{\gamma}+\sigma_e)dW_t = (\frac{\theta}{\gamma}+\sigma_e)\,d\widehat{W}_t - \left(\frac{\theta^2}{\gamma} + \sigma_e^2 + \frac{\sigma_e\theta}{\gamma} + \theta\sigma_e\right)dt$$

and

$$\frac{dZ_t}{Z_t} = [\frac{r-\beta}{\gamma} + \mu_e + \frac{1-\gamma}{2\gamma^2}\theta^2 - \theta\sigma_e]dt + (\frac{\theta}{\gamma}+\sigma_e)d\widehat{W}_t\,.$$

The proposition holds from the last decomposition and the remark that

$$\mu_Z = \frac{r-\beta}{\gamma} + \mu_e + \frac{1-\gamma}{2\gamma^2}\theta^2 - \theta\sigma_e = B^{-1} - A^{-1}$$

Taking the expectation with respect to this new probability measure, we easily prove that

$$\frac{V_0(\nu)}{\varepsilon} = \sup_{\tau\in\mathcal{T}} E[\frac{H_\tau e_\tau}{\varepsilon}(AZ_\tau - B)^-] = \sup_{\tau\in\mathcal{T}} E_{Q^e}[e^{-\frac{\tau}{B}}(AZ_\tau - B)^-] \quad \square$$

Resolution of the optimal stopping problem

From the convexity and monotonicity of the value function $\widehat{V}$, the optimal stopping time is to be chosen in the class of entrance times $\tau(\alpha)$ such that

$$\tau(\alpha) = \inf\{t; Z_t \leq \alpha\}$$

Let us remark that if $\alpha \geq z$, then $\tau(\alpha) = 0$.
The reward associated with such a stopping time is given by

$$\Psi(z,\alpha) = E_{Q^e}(e^{-\tau(\alpha)/B}(AZ_{\tau(\alpha)} - B)^-) = (A(\alpha \wedge z) - B)^- E_{Q^e}(e^{-\tau(\alpha)/B})$$

The computation of such an expression follows from the well known closed formula for the Laplace transform of the law of a Brownian entrance time : let $T(b,\mu) = \inf\{t|\, \mu t + \widehat{B}_t = b\}$, be the entrance time for the generalized Brownian motion $(\widehat{B}_t + \mu t, t \geq 0)$.

$$E(e^{-\lambda T(b,\mu)} \mathbb{1}_{T(b,\mu)<\infty}) = \exp\Big(b\mu - |b|\sqrt{\mu^2 + 2\lambda}\Big), \qquad \lambda > 0. \tag{4.7}$$

The process $\ln(Z)$ is a Brownian motion with drift ∇ where

$$\begin{aligned} \nabla &= \mu_Z - \frac{1}{2}(\sigma_Z)^2 \\ &= \frac{r - \beta - \theta^2/2}{\gamma} + \mu_e - \frac{\sigma_e^2}{2} - \theta\sigma_e - \frac{\theta\sigma_e}{2\gamma}, \end{aligned}$$

Therefore, the stopping time $\tau(\alpha)$ may be seen as a Brownian entrance time, that is,

$$\tau(\alpha) = T\left(\frac{1}{\sigma_Z}\ln(\frac{\alpha}{z}), \frac{\nabla}{\sigma_Z}\right), \quad \text{for } \alpha < z.$$

The formula (4.7) with $\mu = \dfrac{\nabla}{\sigma_Z}$, $b = \dfrac{1}{\sigma_Z}\ln\left(\dfrac{\alpha}{z}\right)$ and $\lambda = \frac{1}{B}$ provides

$$E_{Q^e}(\exp -\frac{\tau(\alpha)}{B}) = \left(\frac{\alpha \wedge z}{z}\right)^{\Delta} \tag{4.8}$$

with

$$\Delta = \frac{1}{(\sigma_Z)^2}[\nabla + \sqrt{\nabla^2 + 2\sigma_Z^2 B^{-1}}] \tag{4.9}$$

The optimal stopping time is associated with the value of α which maximizes $\Psi(z,\alpha)$, that is, since $\Delta > 0$, with $\alpha^* \wedge z$, where

$$\alpha^* = \frac{\Delta B}{A(1+\Delta)}$$

and we obtain

$$\widehat{V}(z) = \Psi(z, \alpha^* \wedge z) = (B - A(\alpha^* \wedge z))^+ \left(\frac{\alpha^* \wedge z}{z}\right)^{\Delta}$$

It is easy to check that $\alpha^* \leq 1$.
Let us come back to the previous problem, and summarize the results concerning the dual constrained problem in the following theorem

Theorem 4.2 *The closed formula for the optimal wealth associated with the constrained dual problem is provided from the free boundary of this problem*

$$\begin{aligned} b(\varepsilon) &= \Big(\frac{A(1+\Delta)}{\Delta B\varepsilon}\Big)^{\gamma} \\ Y_0(\nu) &= 0 \quad \textit{if} \quad \nu \geq b(\varepsilon) \\ \\ &= \frac{B\varepsilon}{1+\Delta}\Big(\frac{\Delta B\varepsilon\nu^{\frac{1}{\gamma}}}{A(1+\Delta)}\Big)^{\Delta} + A\nu^{-\frac{1}{\gamma}} - B\varepsilon \quad \textit{otherwise} \end{aligned}$$

PROOF: The form of $Y_0(\nu)$ follows from the equality $Y_0(\nu) = X_0^f(\nu)+V_0(\nu)$ (Check that $Y_0(\nu) \geq 0$).
Therefore, the boundary $b(t) = \inf\{\nu|\, Y_t(\nu) = 0\}$ is

$$b(t) = \Big(\frac{A(1+\Delta)}{\Delta B e_t}\Big)^{\gamma}.$$

□

This allows us to obtain the solution of the primal problem

Theorem 4.3 *The optimal consumption function is given by* $c^*(\nu_x) = (\nu_x \wedge b(\varepsilon))^{-1/\gamma} = \varepsilon(z \vee \alpha^*)$ *where* $z = \nu_x^{-1/\gamma}\varepsilon^{-1}$. *The optimal consumption and optimal wealth are linked in a feedback formula*

$$x^* = \varepsilon\Big(\frac{B}{1+\Delta}\Big(\frac{\Delta B}{A(1+\Delta)}\frac{\varepsilon}{c^*}\Big)^{\Delta} + A\frac{c^*}{\varepsilon} - B\Big)$$

Associated with an initial wealth equal to 0, *the maximal value for the consumption function is an* α^**-fraction of the income flow, where* $\alpha^* = \dfrac{\Delta B}{A(1+\Delta)}$ *is strictly smaller than the fraction* $\dfrac{B}{A}$ *associated with the free problem.*

PROOF: The value ν_x is such that $Y_0(\nu_x) = x$ and, if $\sigma = T \wedge \inf\{t : H_t\nu_x = b(t)\}$, we have

$$D_t^* = 1, t \leq \sigma,\ D_t^* = \inf_{\sigma \leq s \leq t} \frac{1}{H_s^{\sigma}}(\frac{e_\sigma}{e_s})^{1-\gamma}$$

The optimal consumption follows

$$\frac{dc_t^*}{c_t^*} = \frac{1}{\gamma-1}\Big(\frac{dD_t^*}{D_t^*} - [r + \frac{1}{2}\theta^2\frac{\gamma-2}{\gamma-1}]dt - \theta\, dW_t\Big)$$

□

We present, in figure 1, the curve c^*/ε as well as the straigth line solution of the free problem in the plane $(x^*/\varepsilon, c^*/\varepsilon)$ for $r = 0.1, b = 0.2, \sigma = 0.1, \mu_e = 1, \sigma_e =$

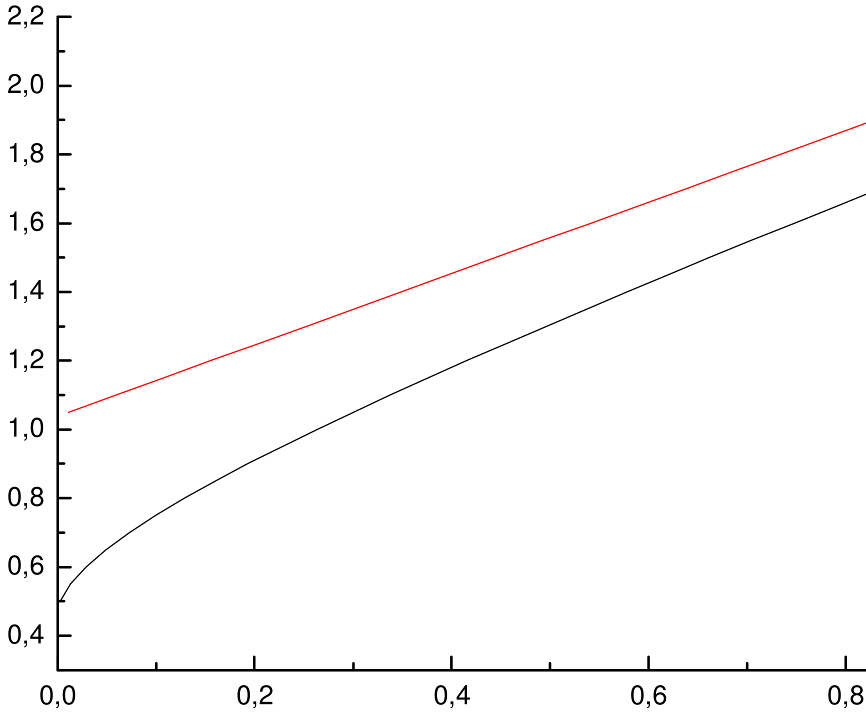

Figure 1.

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5 Appendix

5.1 Sketch of the proof of Theorem 2.1

Uniqueness: It follows from Itô's lemma. Indeed, suppose that Y_1, K_1, π_1 and Y_2, K_2, π_2 are two solutions of (2.9) and denote by Y, K, π the differences $Y_1 - Y_2, \ldots$. Then, Itô's formula applied to Y^2 provides

$$E\Big(Y_t^2 + \int_t^T ||\pi_s||^2 ds\Big) = 2\,E\Big(-\int_t^T r_s Y_s^2\, ds + \int_t^T Y_s dK_s\Big)$$

Since $\int_t^T Y_s dK_s = \int_t^T -Y_{2,s}dK_{1,s} - Y_{1,s}dK_{2,s}$, the right member of the preceeding equality is negative and the uniqueness follows.

Existence: Let us consider $X^{c,\zeta}$ the free-value process of the consumption plan (c,ζ), and define the price $Z^{c,\zeta}$ of an American option with payoff $(X^{c,\zeta})^-$, that is,

$$Z_t^{c,\zeta} \stackrel{def}{=} \operatorname*{ess\ sup}_{\tau\in\mathcal{T}(t)} E\Big[(X_\tau^{c,\zeta})^- |\mathcal{F}_t\Big].$$

The theory of American option and the optimal stopping times provides[6] that

$$\left\{\begin{array}{rcl} dZ_t^{c,\zeta} & = & r_t Z_t^{c,\zeta}\, dt - dK_t^{c,\zeta} + \pi_t^{c,\zeta}(dW_t + \theta_t\, dt), \qquad Z_T^{c,\zeta} = 0, \\ Z_t^{c,\zeta} & \geq & (X_t^{c,\zeta})^-, ,\ \forall t \leq T \\ \text{where} & K & \text{is an continuous increasing process flat outside } \{t\,|Z_t^{c,\zeta} > (X_t^{c,\zeta})^-\} \end{array}\right. \tag{5.1}$$

Moreover, it is optimal to stop at the first time where $Z^{c,\zeta} = (X^{c,\zeta})^-$.

Let us define $Y_t^{c,\zeta} = Z_t^{c,\zeta} + X_t^{c,\zeta}$. Then

$$Y_t^{c,\zeta} \stackrel{def}{=} \operatorname*{ess\ sup}_{\tau\in\mathcal{T}(t)} E\Big(\int_t^\tau H_s^t(c_s - e_s)\, ds + H_T^t\, \zeta\, 1\!\!1_{\tau=T}|\mathcal{F}_t\Big)$$

We deduce from the equation (2.9) that $(Y_t^{c,\zeta}, t\in[0,T])$ is a superstrategy against (c,ζ), and from the properties of the increasing process K, that $(Y_t^{c,\zeta}, t\in[0,T])$ is the minimal superhedging strategy, i.e., by definition the fair value. The properties of the Snell envelope imply that the stopping time $\sigma = \inf\{s\,|Y_s^{c,\zeta} = 1\!\!1_{s=T}\zeta\}$ is optimal, i.e.,

$$Y_0^{c,\zeta} = E(\int_0^\sigma H_s(c_s - e_s)\, ds + H_T\zeta 1\!\!1_{\sigma=T})\,.$$

In order to prove the characterisation of affordable plans by means of decreasing procsses, it suffices to prove that

$$Y_0^{c,\zeta} = \sup_{D\in\mathcal{D}} E\Big(\int_0^T H_s D_s(c_s - e_s)\, ds + H_T D_T^- \zeta\Big)\,.$$

[6]The difficulty is to establish the square integrability of the processes $Z^{c,\zeta}$ and $\pi^{c,\zeta}$. This is done in [8].

Let τ be a stopping time such that $\tau \leq T$, and define a sequence of decreasing processes by

$$D^n_s = \exp -(\int_0^s n \mathbb{1}_{\tau \leq u}\, du) \qquad s < T,\ D^n_T = 0$$

Since $D^n_s \to \mathbb{1}_{s \leq \tau}$ and $D^n_{T^-} \to \mathbb{1}_{\tau = T}$, the following inequality holds

$$\sup_{D \in \mathcal{D}} E\Big(\int_0^T H_s D_s (c_s - e_s)\, ds + H_T D^-_T \zeta\Big) \leq \sup_{\tau \in \mathcal{T}} E\Big(\int_0^\tau H_s (c_s - e_s)\, ds + H_T\, \zeta\, \mathbb{1}_{\tau = T}\Big)$$

The reverse inequality is proved using the right-continuous inverse of D, $\Gamma_t = \inf\{s \,|\, D_s \leq t\}$. For each t, Γ_t is a stopping time. The set-equality $\{D_s > u\} = \{\Gamma_u > s\}$ implies that

$$\int_0^T H_s D_s (c_s - e_s)\, ds + H_T D^-_T \zeta = \int_0^1 du [\int_0^{\Gamma_u} H_s (c_s - e_s)\, ds + H_T\, \zeta\, \mathbb{1}_{\Gamma_u = T}]$$

and the result follows. The same method leads to

$$Y^{c,\zeta}_t = \mathrm{ess} \sup_{D \in \mathcal{D}, D_t = 1} E(\int_t^T H^t_s D_s (c_s - e_s)\, ds + H^t_T D^-_T \zeta | \mathcal{F}_t). \qquad \square$$