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# FRACTIONAL INTEGRALS AND BROWNIAN PROCESSES

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## I. Introduction

The aim of this paper is to give a pure analytic viewpoint of the regularity of certain random processes. Instead of looking for regularity of a given type, as is usually done in probability, we consider a convenient Banach space of Hölder continuous functions : the classical Liouville space $\mathcal{J}_{\alpha,p}$. This is the range of $L^p$ under the Liouville fractional primitive operator

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds$$

Then it is straightforward to check that a process satisfying the hypothesis of Kolmogorov criterion is an $\mathcal{J}_{\alpha,p}$-valued $L^p$-function.
In the case of quasi-sure analysis on the Wiener space, the Kolmogorov criterion obviously extends by this way into a property of $\mathcal{J}_{\alpha,p}$-valued $W^{r,p}$-functions.

In a second part we give as applications very simple proofs of the existence of different $\alpha$-fractional Brownian motions, with $\frac{1}{2} \le \alpha \le 1$. Notice that the Wiener measure is obtained for $\alpha = 1$. The so-called classical $\alpha$-fractional Brownian motion is associated to a Cameron-Martin space which is a fractional Sobolev space image of $L^2(\mathbb{R}, dx)$ by a translate (due to the absence of integrability) of the Riesz potentiel of order $\alpha$. Here we see that the image of $L^2([0,1], dx)$ by the Liouville operator gives rise to a fractional Brownian motion which is easier to handle than the classical one, though it has a very complicated covariance.

In the following section we deal with multiparameter processes. The natural extension of the Liouville space gives rise to a non classical kind of Hölder continuity, and to a more appropriate version of the Kolmogorov lemma. This turns out to be a good method for defining fractional Brownian sheets and also their regularity.

A serious problem (for those who work in financial probabilities, *cf.* [3]) is the fractional stochastic calculus. A first step in this direction is to start with by defining the Wiener integral. We show that if $f$ (resp. $g$) is $\alpha$ (resp.$\beta$) Hölder continuous, then we can define $\int f dg$ by a Riemman sum if $\alpha + \beta > 1$. From this result we get $\int f dX_t^\alpha$, where $X_t^\alpha$ stands for our fractional Brownian motion, when $f$ is $\beta$-Hölder with $\alpha + \beta > 1$. Note that $X_t^\alpha$ is only $(\alpha - \frac{1}{2})$-Hölder, so that the stochastic result needs a weaker hypothesis than the deterministic one.

For completeness sake, in the last paragraph we compare the Liouville space with the Slobodetzki space, which is another classical Banach space of Hölder continuous functions.

## II. The Liouville space.

Let $I = [0,1]$, $p \in ]1, +\infty[$, $0 < \alpha \leq 1$ and (*cf.* for example [6])

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} f(x-t)dt$$

be the primitive of order $\alpha$ (Liouville integral). As $x_+^{\alpha-1} = (x \vee 0)^{\alpha-1}$ is locally integrable, the range $I^\alpha(L^p(I,dx))$ is included in $L^p(I,dx)$. As $I^\alpha$ is one to one, $N_{\alpha,p}(I^\alpha f) = N_p(f)$ defines a norm on the range $\mathcal{J}_{\alpha,p} = I^\alpha(L^p)$. Obviously $\mathcal{J}_{\alpha,p}$ is a separable Banach space. The definition formally extends to the case $p = +\infty$.

Recall the very well known following facts : the map $\alpha \to I^\alpha f$ extends into an entire function with values in $\mathcal{D}'$ (the space of distributions), $I^\alpha f$ is said to be the "fractional primitive" of $f$, $D^\alpha f = I^{-\alpha} f$ is the "fractional derivative", and one has $I^{\alpha+\beta} f = I^\alpha I^\beta f$.

**1 Proposition :** *for $\alpha > \beta$ and $p > 1$, the embedding $\mathcal{J}_{\alpha,p} \subset \mathcal{J}_{\beta,p}$ is compact.*

Proof : it suffices to prove that $I^{\alpha-\beta} : L^p(I) \to L^p(I)$ is a compact operator. Indeed this is the convolution operator with $u_{\alpha-\beta}(x) = x_+^{\alpha-\beta-1}/\Gamma(\alpha-\beta)$ which is integrable over $I$.

## III. Comparison with spaces of Hölder continuous functions.

Denote $\mathcal{H}_\alpha$ the space of $\alpha$-Hölder continuous functions vanishing at 0 with its natural norm. This is not a separable space. Nevertheless we have

**2 Proposition :** *for $\alpha > 1/p \geq 0$ and $\beta > \gamma \geq 0$, both following inclusions hold, and the last one is compact*

$$\mathcal{J}_{\alpha,p} \subset \mathcal{H}_{\alpha-1/p} \quad \& \quad \mathcal{H}_\beta \subset \mathcal{J}_{\gamma,\infty}$$

Proof : put $q = p/(p-1)$, and write $I^\alpha f(x) = f * u_\alpha(x)$ with $u_\alpha(x) \in L^q(I)$ for $\alpha > 1/p$. Let $h > 0$, and put $u_\alpha^h(x) = u_\alpha(x-h)$. This yields

$$|I^\alpha f(x) - I^\alpha f(x-h)| \leq N_p(f) N_q(u_\alpha^h - u_\alpha)$$

$$N_q(u_\alpha^h - u_\alpha)^q \leq c_{\alpha,p} \int_0^{+\infty} |x^{\alpha-1} - (x-h)_+^{\alpha-1}|^q dx \leq C_{\alpha,p} h^{q(\alpha-1/p)}$$

For the last inclusion, take $f \in \mathcal{H}_\beta$, and compute $D^\gamma f = I^{-\gamma} f$ by analytic continuation

$$D^\gamma f(x) = \frac{1}{\Gamma(-\gamma)} \int_0^x t^{-\gamma-1} \left[f(x-t) - f(x)\right] dt + \frac{x^{-\gamma} f(x)}{\Gamma(1-\gamma)} = h(x) + \frac{x^{-\gamma} f(x)}{\Gamma(1-\gamma)}$$

which makes sense for $\beta > \gamma > 0$. This yields $\|D^\gamma f\|_\infty \leq K \|f\|_{\mathcal{H}_\beta}$.
For the compacticity, put $\beta = \gamma + 3\varepsilon$, choose $p > 1/\varepsilon$, write
$\mathcal{H}_\beta \subset \mathcal{J}_{\gamma+2\varepsilon,p} \subset \mathcal{J}_{\gamma+\varepsilon,p} \subset \mathcal{J}_{\gamma,\infty}$ and apply proposition 1.

**3 Remarks :** a) roughly speaking, this means that Hölder continuous functions are those functions wich have fractional derivatives.
b) in the last proof, we can prove that $D^\gamma f$ is continuous by varying $\gamma$.

## IV. Vector valued functions.

Let $B$ be a Banach space endowed with norm $|.|$. In the same way as in the real case, define $\mathcal{J}_{\alpha,p}(B)$ as the space of those functions $f : I \to B$ which can be written

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x g(t)(x-t)^{\alpha-1}\, dt$$

where $g \in L^p(I, B)$. The same properties (except compacticity) as above hold. In our case, we can define $\mathcal{J}_{\alpha,p}(L^p(\Omega,\mu))$. We have the following

**4 Proposition :** *$\mathcal{J}_{\alpha,p}(L^p(\Omega,\mu))$ is canonically isometrically isomorphic with $L^p(\Omega,\mu,\mathcal{J}_{\alpha,p})$ $(1 < p < +\infty)$.*

Proof : it suffices to check that the two norms agree on functions $f : \Omega \to \mathcal{J}_{\alpha,p}$ with finite rank.

## V. The Kolmogorov lemma revisited.

In fact, the last proposition turns out to be the functional-analysis expression of the Kolmogorov lemma. More precisely, let $(X_t)_{t\in I}$ be a process satisfying

$$N_p(X_t - X_s) \leq c|t-s|^{\alpha - \frac{1}{2}}$$

for $\alpha \in ]\frac{1}{2}, 1]$. Applying the preceding proposition yields

**5 Theorem :** *(Kolmogorov) for $\alpha - 1/2 > \beta > 1/p > 0$, this process has a modification with $(\beta - 1/p)$-Hölder continuous trajectories.*

Proof : $X - X_0$ belongs to $\mathcal{H}_{\alpha-\frac{1}{2}}(L^p) \subset \mathcal{J}_{\beta,p}(L^p) \approx L^p(\mathcal{J}_{\beta,p}) \subset L^p(\mathcal{H}_{\beta-1/p})$.

Now assume that $(\Omega, \mu)$ is a Gaussian vector space, and let $W^{r,p}(\Omega, \mu)$ be the $(r, p)$ Sobolev space endowed with the norm $\|f\|_{r,p} = N_p\left((I - L)^{r/2} f\right)$ where $L$ is the Ornstein-Uhlenbeck operator. In view of the above proposition, we obviously get $\mathcal{J}_{\alpha,p}(W^{r,p}(\Omega, \mu)) \approx W^{r,p}(\Omega, \mu, \mathcal{J}_{\alpha,p})$.

Recall [4] that the capacity $c_{r,p}(g)$ for an l.s.c. function $g \geq 0$ on $\Omega$ is defined by

$$c_{r,p}(g) = \mathrm{Inf}\{ \ \|f\|_{r,p} \ \big| \ f \geq g\}$$

and, for every $h$,

$$c_{r,p}(h) = \mathrm{Inf}\{ \ c_{r,p}(g) \ \big| \ g \text{ l.s.c.} \geq |h|\}$$

**6 Definition :** *we say that a process $Y_t$ is a strong modification of $X_t$ if for every $t$ one has $\mu(\{Y_t \neq X_t\}) = 0$ and $Y_t$ is $c_{r,p}$-quasi-continuous.*

**7 Theorem :** *assume that $(X_t)$ satisfies $\|X_t - X_s\|_{r,p} \leq c|t - s|^{\alpha-\frac{1}{2}}$ for $\alpha - 1/2 > \beta > 1/p$. Then $X_t$ has a strong modification $Y_t$ with $(\beta - 1/p)$-Hölder continuous trajectories.*
*In addition, for every $\varepsilon > 0$ there exists a compact $K \subset \Omega$ such that $c_{r,p}(\Omega \backslash K) < \varepsilon$ and such that $(t, \omega) \to Y_t(\omega)$ is continuous on $K$ and uniformly Hölder in $t$.*

Proof : in fact we have

$$X - X_0 \in \mathcal{H}_{\alpha-\frac{1}{2}}(W^{r,p}) \subset \mathcal{J}_{\beta,p}(W^{r,p}) \approx W^{r,p}(\mathcal{J}_{\beta,p})$$

We have $W^{r,p}(\mathcal{J}_{\beta,p}) \subset \mathcal{L}^1(c_{r,p}, \mathcal{J}_{\beta,p})$. The last space is included in $\mathcal{L}^1(c_{r,p}, \mathcal{H}_{\beta-1/p})$, so the proof is complete (*cf.* [4]).

**8 Corollary :** *(see also* [1] *and* [2]*) let $(\Omega, \mu)$ be the Wiener space, and let $X_t = \int_0^t u_s dW_s$ be a stochastic integral where $u_s$ is a predictable process belonging to $L^p(I, dt, W^{r,p}(\Omega, \mu))$. Then for $1/2 - 1/p > \beta > 1/p$, the process $X$ belongs to $L^p(\Omega, \mu, \mathcal{J}_{\beta,p})$, and then $c_{r,p}$-quasi-every trajectory is Hölder continuous.*

Proof : write

$$Y = (I - L)^{r/2}(X_b - X_a) = \int_a^b \left((2I - L)^{r/2} u_t\right) dW_t$$

By Burkholder's inequality, we get

$$N_p(Y) \leq c \left[\int_a^b N_p(v_t)^2 \, dt\right]^{\frac{1}{2}}$$

with $v_t = (I-L)^{r/2}u_t$. This yields

$$N_p(Y) \leq c|b-a|^\gamma \left[\iint |v_t(\omega)|^p \, dt \, d\omega\right]^{1/p}$$

with $\gamma = 1/2 - 1/p$. ∎

**9 Remark :** if $t \to u_t$ is bounded with values in $W^{r,p}$ (*i.e.* belongs to $L^\infty(W^{r,p})$, then we can get better estimates ($1/2 > \beta > 1/p$).

## VI. Fractional Brownian motions

### A general setup

Let $\mu^0$ be the white noise measure on $L^2(I, dx)$ or on $L^2(\mathbb{R}, dx)$. Let $V : L^2 \to \mathcal{C}$ an injective continuous linear mapping. Note $\mu = V(\mu^0)$ the image (pro-) measure (or cylindrical measure) on $\mathcal{C}$. Denote $W_t$ the evaluation at $t$.

**10 Proposition :** *assume $\alpha \in ]\frac{1}{2}, 1]$. Then the following are equivalent*
*a) the estimate holds*

$$\mathbb{E}_\mu\left((W_t - W_s)^2\right) \leq c|t-s|^{2\alpha-1} \tag{1}$$

*b)* $V(L^2) \subset \mathcal{H}_{\alpha-\frac{1}{2}}$.

Proof : first inequality makes sense for a (pro-) measure $\mu$ on $\mathcal{C}$, because the function to be integrated is cylindrical. Now, for every $t$, there exists $K_t \in L^2$ such that $W_t(Vf) = \langle K_t, f\rangle_{L^2}$. We get

$$\mathbb{E}_\mu\left((W_t - W_s)^2\right) = N_2(K_t - K_s)^2$$

$$|Vf(t) - Vf(s)|^2 = \langle f, K_t - K_s\rangle^2 \leq cN_2(f)^2|t-s|^{2\alpha-1} \tag{2}$$

Then $V(L^2) \subset \mathcal{H}_{\alpha-\frac{1}{2}}$. Conversely, by the closed graph theorem, the map $V : L^2 \to \mathcal{H}_{\alpha-\frac{1}{2}}$ is continuous, so there exists $c$ such that (2) holds, and then (1) holds.

Note that the (pro-) measure $\mu$ is the canonical (pro-) measure of the Hilbert space $V(L^2)$ endowed with the norm $\|Vf\| = N_2(f)$.
Now we can get the following result which follows from the Kolmogorov lemma

**11 Theorem :** *for $\beta < \alpha - 1/2$ the (pro-) measure $\mu$ extends into a Borel Gaussian measure on $\mathcal{J}_{\beta,p}$.*

Proof : let $\Omega \supset V(L^2)$ be a carrying space for a Borel extension of $\mu$. The map $t \to W_t \in V(L^2)'$ extends into a map $t \to W_t \in \cap_p L^p(\Omega,\mu)$, which satisfies the Kolmogorov condition.

**12 Remark :** putting $X_t = W_t[V(\dot\omega)]$, one gets $X_t(\omega) = \int_0^t K(s,t)dW_s(\omega)$ (Wiener integral), so the law of the process $X_t$ under $\mu$ is the law of $W_t$ under $\mu^\alpha$.

**Liouville fractional Brownian motion.**

Take $V = I^\alpha$. Note $\mu^\alpha = I^\alpha(\mu^0)$. For $\alpha > \frac{1}{2}$, $I^\alpha$ is Hilbert-Schmidt from $L^2$ to $L^2$, so that $\mu^\alpha$ extends into a measure on $L^2$, and we can take $\Omega = L^2$. Note that $\mu^1$ is the Wiener measure. An easy computation yields

$$\mathbb{E}_\alpha(W_t^2) = \int_\Omega W_t^2 d\mu^\alpha = \frac{t^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)^2}$$

The covariance kernel is given by

$$\mathrm{Cov}_\alpha(W_t, W_s) = \frac{1}{\Gamma(\alpha)^2}\int_0^{s\wedge t}(s-u)^{\alpha-1}(t-u)^{\alpha-1}du$$

which is not an elementary function. Nevertheless, we have the estimates

$$\Gamma(\alpha)^2\mathbb{E}_\alpha\left((W_t - W_s)^2\right) \leq \int_{-\infty}^{+\infty}[t_+^{\alpha-1} - (|y-x|+t)_+^{\alpha-1}]^2\,dt$$

$$\mathbb{E}_\alpha\left((W_t - W_s)^2\right) \leq c_\alpha|t-s|^{2\alpha-1}$$

Then according to last theorem, for $1/p < \beta < \alpha - 1/2$ $\mu^\alpha$ is carryied by $\mathcal{J}_{\beta,p} \subset \mathcal{H}_{\beta-1/p}$.

**13 Remarks :** a) For $\alpha = 1$ we have obttain an alternative construction of the Wiener measure.
b) Note that $I^\alpha$ is a semi-group family of operators verifying $I^\alpha(\mu^\beta) = \mu^{\alpha+\beta}$.

**Fourier fractional Brownian bridge.**

Now we deal with kernels

$$K_\alpha(x,y) = \sum_{n\geq 1}\frac{s_n(x)s_n(y)}{\pi^\alpha n^\alpha}$$

where $s_n(t) = \sqrt{2}\sin \pi n t$. For $f \in L^2(I, dx)$, put

$$U^\alpha f(x) = \int_0^1 K_\alpha(x,y) f(y) dy$$

The operator $U^\alpha$ is the (symmetric) fractional power of $U^1$. One verifies that

$$K_2(x,y) = x \wedge y - xy$$

so $\mu^1 = U^1(\mu^0)$ is the Brownian bridge measure. For $\alpha \leq 1$ put $\mu^\alpha = U^\alpha(\mu^0)$.
As above, $U^\alpha$ is Hilbert-Schmidt for $\alpha > \frac{1}{2}$.
Now the Cameron-Martin space $U^\alpha(L^2)$ is a classical fractional Sobolev space.
By the same arguments as above, we are led to compute

$$\mathrm{Cov}_\alpha(W_s, W_t) = \int_\Omega W_s W_t d\mu^\alpha = K_{2\alpha}(s,t)$$

$$\mathbb{E}_\alpha |W_t - W_s|^2 = \sum_{n \geq 1} \frac{|s_n(s) - s_n(t)|^2}{\pi^{2\alpha} n^{2\alpha}} \leq 2K_{2\alpha}\left(\frac{s-t}{2}, \frac{s-t}{2}\right)$$

For $\gamma = 2\alpha > 1$ and small $x$ we have

$$K_\gamma(x,x) = \int_0^\infty \frac{t^{\gamma-1}\,dt}{\pi^\gamma \Gamma(\gamma)} \sum_{n \geq 1} \mathrm{e}^{-nt} s_n(x)^2 = \frac{\sin^2 \pi x}{\pi^\gamma \Gamma(\gamma)} \int_0^\infty \frac{\mathrm{e}^t + 1}{\mathrm{e}^t - 1} \frac{t^{\gamma-1}\,dt}{\mathrm{Cosh}\; t - \cos 2\pi x}$$
$$\leq C_\gamma x^2 \int_0^\infty (1 + \frac{2}{t}) \frac{t^{\gamma-1}\,dt}{t^2 + x^2} \leq C'_\gamma x^{\gamma-1}$$

Then $$\mathbb{E}_\alpha |W_t - W_s|^2 \leq C'_\alpha |t-s|^{2\alpha-1}$$

Then for $1/p < \beta < \alpha - 1/2$, $\mu^\alpha$ is carried by $\mathcal{J}_{\beta,p} \subset \mathcal{H}_{\beta - 1/p}$.

**14 Remark :** Again we obtained a semi-group family of operators $U^\alpha$ satisfying $U^\alpha(\mu^\beta) = \mu^{\alpha+\beta}$. But here the $U^\alpha$ are symmetric operators.

**Fourier fractional Wiener measure**.

Here we deal with kernels

$$K_\alpha(x,y) = \sum_{n \geq 0} \frac{s_n(x) s_n(y)}{\pi_n^\alpha}$$

where $\pi_n = (n + \frac{1}{2})\pi$ and $s_n(t) = \sqrt{2}\sin \pi_n t$. For $f \in L^2(I, dx)$, put

$$U^\alpha f(x) = \int_0^1 K_\alpha(x, y) f(y) dy$$

The operator $U^\alpha$ is the (symmetric) fractional power of $U^1$. As above we easily check that

$$K_2(x, y) = x \wedge y$$

so that $\mu^1 = U^1(\mu^0)$ is the Wiener measure. For $\alpha \leq 1$ put $\mu^\alpha = U^\alpha(\mu^0)$.
Note that $U^\alpha$ is Hilbert-Schmidt for $\alpha > \frac{1}{2}$.
Now the Cameron-Martin space is $U^\alpha(L^2)$, and this is a classical fractional Sobolev space.
By the same arguments as above, we are led to compute

$$\begin{aligned} &\mathrm{Cov}\ (W_s, W_t) = K_{2\alpha}(s, t) \\ &\mathbb{E}_\alpha |W_t - W_s|^2 = \sum_{n \geq 0} \frac{|s_n(s) - s_n(t)|^2}{\pi_n^{2\alpha}} \leq 2K_{2\alpha}\left(\frac{s-t}{2}, \frac{s-t}{2}\right) \end{aligned}$$

For $\gamma = 2\alpha > 1$ and small $x$ we get

$$\begin{aligned} K_\gamma(x, x) &= \frac{1}{\pi^\gamma \Gamma(\gamma)} \int_0^\infty t^{\gamma - 1} dt \sum_{n \geq 0} \mathrm{e}^{-t/2}\, \mathrm{e}^{-nt}\, s_n(x)^2 \\ &\leq \frac{1}{\pi^\gamma \Gamma(\gamma)} \int_0^\infty t^{\gamma - 1} dt \sum_{n \geq 1} \mathrm{e}^{-nt/2}(1 - \cos \pi n x) \leq C''_\gamma |x|^{\gamma - 1} \end{aligned}$$

Hence
$$\mathbb{E}_\alpha |W_t - W_s|^2 \leq c_\alpha |t - s|^{2\alpha - 1}$$

Then the Fourier-Wiener fractional measure $\mu^\alpha$ is carried by the space $\mathcal{J}_{\beta,p}$.

**15 Remarks :** a) Once more we have obtained a semi-group family of operators $U^\alpha$ satisfying $U^\alpha(\mu^\beta) = \mu^{\alpha+\beta}$.
b) For $\alpha = 1$ we have obtained another construction of the Wiener measure.

**Bessel fractional Brownian motion**.

On $\mathbb{R}$, consider the fractional $\lambda$-Bessel potential kernel

$$G_\lambda^\alpha(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2 - 1}\, \mathrm{e}^{-\lambda t}\, g_t(x) dt$$

where
$$g_t(x) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t})$$
is the density of the classical Gaussian semi-group.
For $f \in L^2(\mathbb{R}, dx)$, put $V_\lambda^\alpha f(x) = \int G_\lambda^\alpha(x-y)f(y)dy$. For $\lambda > 0$, we obtain a semi-group family of operators $(V_\lambda^\alpha)_{\alpha \geq 0}$ on $L^2(\mathbb{R}, dx)$.
We denote $\mu_\lambda^\alpha$ the image Gaussian (pro)-measure $V_\lambda^\alpha(\mu^0)$. The Cameron-Martin space is $V_\lambda^\alpha(L^2)$, and it is well known that it does not depend on $\lambda > 0$. This is again a classical fractional Sobolev space $H^\alpha(\mathbb{R})$, and it is included in $\mathcal{C}(\mathbb{R})$ for $\alpha > \frac{1}{2}$.
Then we may consider
$$\mathrm{Cov}_\alpha(W_x, W_y) = \int G_\lambda^\alpha(x-z)G_\lambda^\alpha(z-y)dz$$
and compute
$$\begin{aligned}\mathbb{E}_\alpha|W_x - W_y|^2 &= N_2(\Phi_0 - \Phi_u)^2 = 2\pi N_2(\widehat{\Phi}_0 - \widehat{\Phi}_u)^2 \\ &= 2\pi \int \frac{\sin^2 ux/2}{(\lambda + x^2)^\alpha} dx \leq 2\pi \int \frac{\sin^2 ux/2}{|x|^{2\alpha}} dx = c_\alpha |u|^{2\alpha - 1}\end{aligned}$$
where $u = x - y$, $\Phi_u(z) = G_\lambda^\alpha(u-z)$, $\widehat{\Phi}$ denotes the Fourier transform, and $N_2$ is the norm in $L^2(\mathbb{R}, dz)$. Localizing these results, we get that for $1/p < \beta < \alpha - 1/2$, $\mu_\lambda^\alpha$ lies on the space of $\beta$-Hölder continuous functions on $\mathbb{R}$.

**The classical so-called fractional Brownian motion**.

First, for $\alpha > \frac{1}{2}$, the variance $\mathbb{E}_\alpha|W_x|^2 = \lambda^{\frac{1}{2}-\alpha}t^{2\alpha-1}\Gamma(\alpha - \frac{1}{2})/(\Gamma(\alpha)\sqrt{2\pi})$ converges to $+\infty$ as $\lambda \to 0$. On the other hand, $\mathbb{E}_\alpha[(W_x - W_y)^2] \leq c_\alpha |y-x|^{2\alpha-1}$ remains bounded. Then we replace $\mu_\lambda^\alpha$ with $\nu_\lambda^\alpha = T_0(\mu_\lambda^\alpha)$, where $(T_0(\omega))(x) = \omega(x) - \omega(0)$. The covariance is unchanged, and the variance becomes $\leq c_\alpha |x|^{2\alpha-1}$ which is bounded as $\lambda \to 0$. Every $\nu_\lambda^\alpha$ is carried by the Fréchet space $\mathcal{J}_{\beta,p}^0(\mathbb{R})$ of functions $\omega$ vanishing at 0, and belonging to every $\mathcal{J}_{\beta,p}([-R,R])$ $R > 0$ endowed with the obvious semi-norms $\|\omega\|_{\beta,p,R}$ ; and $1/p < \beta < \alpha - \frac{1}{2}$.

**16 Proposition :** *as $\lambda \to 0$, $\nu_\lambda^\alpha$ narrowly converges to a measure $\nu^\alpha$ on the Fréchet space $\mathcal{J}_{\beta,p}([0,+\infty[)$ for $1/p < \beta < \alpha - \frac{1}{2}$. We have*
$$\mathrm{Cov}_{\nu^\alpha}(W_x, W_y) = c_\alpha(|x|^{2\alpha-1} + |y|^{2\alpha-1} - 2|x-y|^{2\alpha-1})$$
Proof : the canonical injection $\mathcal{J}_{\gamma,p}([0,+\infty[) \subset \mathcal{J}_{\beta,p}([0,+\infty[)$ is compact for $\alpha - 1/ > \gamma > \beta$ and $\int \|\omega\|_{\gamma,p,R}^p d\nu_\lambda^\alpha(\omega)$ is bounded, so the Prokhorov condition is satisfied on $\mathcal{J}_{\beta,p}([0,+\infty[)$. Then $\nu_\lambda^\alpha$ narrowly converges to a measure $\nu^\alpha$ on $\mathcal{J}_{\beta,p}([0,+\infty[)$.

**VII. The double-parameter case.**

For $0 < \alpha, \beta \leq 1$, put

$$I^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y s^{\alpha-1} t^{\beta-1} f(x-s, y-t)\, ds dt$$

Note $\mathcal{J}_{\alpha,\beta,p} = I^{\alpha,\beta}(L^p([0,1]^2))$ endowed with the norm $N_{\alpha,\beta,p}(I^{\alpha,\beta}(f)) = N_p(f)$. We have two obvious canonical isomorphisms of $\mathcal{J}_{\alpha,\beta,p}$ on the two vector valued function spaces $\mathcal{J}_{\alpha,p}(\mathcal{J}_{\beta,p})$ and $\mathcal{J}_{\beta,p}(\mathcal{J}_{\alpha,p})$.

We get the same compact inclusions as in section I :

For $$\begin{cases} 1 \geq \alpha \ > \beta \ > 0 \\ 1 \geq \alpha' > \beta' > 0 \end{cases}$$

then $$\mathcal{J}_{\alpha,\alpha',p} \subset \mathcal{J}_{\beta,\beta',p}$$

Moreover for $\alpha, \alpha' > 1/p \geq 0$, $\beta > \gamma > 0$ and $\beta' > \gamma' > 0$

$$\mathcal{J}_{\alpha,\alpha',p} \subset \mathcal{H}_{\alpha-1/p,\alpha'-1/p} \quad \& \quad \mathcal{H}_{\beta,\beta'} \subset \mathcal{J}_{\gamma,\gamma',p}$$

where $\mathcal{H}_{\beta,\beta'}$ stands for $\mathcal{H}_\beta(\mathcal{H}_{\beta'}) \approx \mathcal{H}_{\beta'}(\mathcal{H}_\beta)$.

Now we are in a position to claim a multiparameter lemma “à la Kolmogorov”

**17 Lemma :** *: let $X_{s,t}$ be a double parameter process satisfying*

$$N_p(X_{s,t} - X_{u,v}) \leq c|s-u|^{\alpha-\frac{1}{2}}|t-v|^{\beta-\frac{1}{2}}$$

*for $\alpha, \beta \in ]\frac{1}{2}, 1]$. Then for $\alpha - 1/2 > \alpha' > 1/p$, $\beta - 1/2 > \beta' > 1/p$, this process has a modification with double Hölder continuous trajectories.*

Proof : as above, write

$$X - X_0 \in \mathcal{H}_{\alpha-\frac{1}{2},\beta-\frac{1}{2}}(L^p) \subset \mathcal{J}_{\alpha',\beta',p}(L^p) \approx L^p(\mathcal{J}_{\alpha',\beta',p}(L^p)) \subset L^p(\mathcal{H}_{\alpha'-1/p,\beta'-1/p})$$

**18 Remark :** in the case of a Gaussian vector space $(\Omega, \mu)$, it is straightforward to get strong modifications for $W^{r,p}$-valued processes. (*cf.* [5]).

**Fractional Brownian sheet.**

As in section VI, put $\mu^{\alpha,\beta} = I^{\alpha,\beta}(\mu^{0,0})$ where $\mu^{0,0}$ is the white noise measure on $L^2(I^2, dxdy)$. We have

$$\mathbb{E}_{\alpha,\beta}(W_{s,t}^2) = \iint W_{s,t}^2 d\mu^{\alpha,\beta} = \frac{s^{2\alpha-1}t^{2\beta-1}}{(2\alpha-1)(2\beta-1)\Gamma(\alpha)^2\Gamma(\beta)^2}$$

As in section VI, with $h = |s-t|,\ k = |x-y|$, we have the estimate

$$\Gamma(\alpha)^2\Gamma(\beta)^2\mathbb{E}_{\alpha,\beta}\left((W_{s,t} - W_{x,y})^2\right) \leq \iint_{\mathbb{R}^2} [(u_+^{\alpha-1} - (h+u)_+^{\alpha-1})(v_+^{\beta-1} - (k+v)_+^{\beta-1})]^2\, dudv$$

$$\mathbb{E}_{\alpha,\beta}\left((W_{s,t} - W_{x,y})^2\right) \leq c_{\alpha,\beta}|s-x|^{2\alpha-1}|t-y|^{2\beta-1}$$

Hence, by the same arguments as in section VI, this yields

$$\mathbb{E}_{\alpha,\beta}(N^p_{\alpha',\beta',p}) = \int_\Omega N^p_{\alpha',\beta',p}(\omega)d\mu^{\alpha,\beta}(\omega) < +\infty$$

for $\alpha' < \alpha - 1/2$ and $\beta' < \beta - 1/2$. Hence $\mu^{\alpha,\beta}$ lies on $\mathcal{J}_{\alpha',\beta',p} \subset \mathcal{H}_{\alpha'-1/p,\beta'-1/p}$ for $\alpha' \wedge \beta' > 1/p$.

## VIII. Application of Liouville spaces to Riemann-Stieltjes sums

**19 Lemma :** *let $f$ and $g$ be $\mathcal{C}^1$-functions, $f(0) = 0$. Then*

$$\left| \int_0^a f(t)dg(t)dt \right| \leq C\|f\|_\alpha \|g\|_\beta a^{1+\varepsilon}$$

*where $\|.\|_\alpha$ stands for the Hölder norm on $[0,1]$, $0 < \varepsilon < \alpha + \beta - 1$.*

Proof : we can assume that $g(a) = 0$. Put $h(t) = g(a-t)$, then $J = \int_0^a f(t)g'(t)dt = (f * h)'(a)$. Choose $\alpha' < \alpha$, $\beta' < \beta$, $\alpha' + \beta' = 1 + \varepsilon$. There exists $\varphi \in L^\infty$, $\psi \in L^\infty$ satisfying $f = I^{\alpha'}\varphi$, $h = I^{\beta'}\psi$. Then

$$|J| = |I^\varepsilon(\varphi * \psi)(a)| \leq ca^{1+\varepsilon}N_\infty(\varphi)N_\infty(\psi) \leq c'a^{1+\varepsilon}N_{\alpha',\infty}(f)N_{\beta',\infty}(h) \leq c''a^{1+\varepsilon}\|f\|_\alpha\|g\|_\beta$$

**20 Corollary :** *let $\pi_n = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$. We have*

$$\left| \int_a^b f(t)dg(t) - \sum_{i=0}^{n-1} f(t_i)\left(g(t_{i+1}) - g(t_i)\right) \right| \leq c\delta^\varepsilon \|f\|_{\alpha'}\|g\|_{\beta'}$$

*where $\delta = \mathrm{Sup}_i\, |t_{i+1} - t_i|$.*

Proof : replacing $f$ with $f - f(t_i)$, we get

$$\left|\int_{t_i}^{t_{i+1}} f dg - f(t_i)(g(t_{i+1}) - g(t_i))\right| \leq c\delta_i^{1+\varepsilon} \|f\|_{\alpha'} \|g\|_{\beta'}$$

$$\left|\sum \int_{t_i}^{t_{i+1}} f dg - f(t_i)(g(t_{i+1}) - g(t_i))\right| \leq c\|f\|_{\alpha'}\|g\|_{\beta'} \operatorname{Sup}_i \delta_i^{\varepsilon} \sum_i \delta_i \leq c'\delta^{\varepsilon} \|f\|_{\alpha'} \|g\|_{\beta'}$$

**21 Theorem :** *let* $f \in \mathcal{H}_\alpha$ *and* $g \in \mathcal{H}_\beta$ *on* $[0, a]$ *with* $\alpha + \beta > 1$, *then*

$$\int_0^a f dg = \operatorname{Lim} \sum_i f(t_i)(g(t_{i+1}) - g(t_i))$$

*converges when the subdivision refines indefinitely.*

Proof : choose $\alpha' < \alpha$, $\beta' < \beta$. Then $f$ (resp. $g$) belongs to the closure of $\mathcal{C}_0^1$ in $\mathcal{H}_{\alpha'}$ (resp. $\mathcal{H}_{\beta'}$). Applying the inequality of the last corollary gives the result.

**22 Proposition :** $F(a) = \int_0^a f dg$ *is Hölder continuous of order* $\beta$.

Proof : $|F(a+h) - F(a) - f(a)(g(a+h) - g(a))|$ is majorized by $C\|f\|_\alpha \|g\|_\beta h^{1+\varepsilon}$, so $|F(a+h) - F(a)| \leq K h^\beta$

**23 Corollary :** $fg = F + G$ *with* $F \in \mathcal{H}_\alpha$, $G \in \mathcal{H}_\beta$.

Proof : by the chain rule it suffices to take $F(x) = \int_0^x f(t)dg(t)$, $G(x) = \int_0^x g(t)df(t)$.

## Application to stochastic Riemann sums

Let $\alpha \in ]\frac{1}{2}, 1[$, and consider the Wiener integral

$$X_t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} dW_s$$

We have $\mathbb{E}(X_t^2) = \dfrac{t^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)^2}$. Let $\varphi$ be a $\beta$-Hölder continuous function with $\alpha + \beta > 1$. Define

$$Z = \int_0^1 (\widetilde{I}^{\alpha-1}(\varphi - \varphi(1))(s) dW_s + \varphi(1) X_1$$

where $\widetilde{I}^{\alpha-1}$ is the adjoint operator of $I^{\alpha-1}$. It is easily checked as in section III that $\varphi - \varphi(1) = \widetilde{I}^{1-\alpha} f$ with $f \in L^2([0,1])$.

**24 Proposition :** *let*

$$\sum_i \varphi(t_i)(X_{t_{i+1}} - X_{t_i})$$

*be Riemann sums. These converge in a weak sense to $Z$, that is,*

$$\mathbb{E}(MZ) = \operatorname{Lim} \mathbb{E}\left(M \sum_i \varphi(t_i)(X_{t_i+1} - X_{t_i})\right)$$

*for every "continuous" Gaussian martingale that is of the form*

$$M_t = \int_0^t \psi(s) dW_s$$

*where $\psi$ is a continuous function on $[0,1]$.*
Proof : write

$$\mathbb{E}\left(M_{t_{i+1}}, \varphi(t_i)(X_{t_{i+1}} - X_{t_i})\right) = \varphi(t_i)[\mathbb{E}\left(M_{t_{i+1}}, X_{t_{i+1}}\right) - \mathbb{E}\left(M_{t_i}, X_{t_i}\right)]$$

On the other hand we get

$$\mathbb{E}\left(M_t, X_t\right) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \psi(s) ds = (I^\alpha \psi(t))$$

Then we get

$$\mathbb{E}\left(M_{t_{i+1}}, \varphi(t_i)(X_{t_{i+1}} - X_{t_i})\right) = \varphi(t_i)[I^\alpha \psi(t_{i+1}) - I^\alpha \psi(t_i)]$$

In view of lemma 19 the sum of these terms converges to

$$\int_0^1 \varphi(s) d(I^\alpha \psi)(s)$$

**25 Proposition :** *let $Y_t^\lambda = \int_0^t \mathrm{e}^{-\lambda(t-s)}\, dW_s$ the Ornstein-Uhlenbeck process of parameter $\lambda > 0$. We have*

$$Z = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{d\lambda}{\lambda^\alpha} \int_0^1 \varphi(s) dY_s^\lambda$$

Proof : this a straightforward computation.

**IX. The Slobodetzki space.**

The Slobodetzki (semi)-norm of a Borel function $f$ on $I = [0,1]$ is defined by

$$\|f\|_{\alpha,p} = \Big[\iint_{I^2} \frac{|f(x)-f(y)|^p}{|x-y|^{1+p\alpha}}dxdy\Big]^{1/p}$$

The Slobodetzki space $\mathcal{S}_{\alpha,p}$ is the closure in this norm of $\mathcal{C}^1$-functions vanishing at 0.

**26 Proposition :** *(Hardy type inequality) for $\alpha > 0$ and $q = p/(p-1)$ there holds*

$$N_p(x^{-\alpha} I^\alpha f) \leq \frac{\Gamma(1/q)}{\Gamma(\alpha + 1/q)} N_p(f)$$

Proof : let $g(x) = \Gamma(\alpha) x^{-\alpha} I^\alpha f(x)$. Then $g = \int_0^1 t^{\alpha-1} f_t dt$ with $f_t(x) = f(x - xt)$

$$N_p(g) \leq \int_0^1 t^{\alpha-1} N_p(f_t)dt \leq N_p(f)\int_0^1 t^{\alpha-1}(1-t)^{-1/p}dt = \frac{\Gamma(\alpha)\Gamma(1/q)}{\Gamma(\alpha+1/q)}N_p(f)$$

**27 Theorem :** *for $1 \geq \alpha > \beta > \gamma > \delta > 0$ and $\beta > 1/p > 0$ the following compact inclusions hold*

$$\mathcal{S}_{\alpha,p} \subset \mathcal{J}_{\beta,p} \subset \mathcal{J}_{\gamma,p} \subset \mathcal{S}_{\delta,p}$$

Proof : first recall that $D^\beta = I^{-\beta}$ can be defined by analytic extension by

$$D^\beta f(x) = \frac{1}{\Gamma(-\beta)}\int_0^x t^{-\beta-1}[f(x-t) - f(x)]dt + \frac{x^{-\beta}f(x)}{\Gamma(1-\beta)} = h(x) + \frac{x^{-\beta}f(x)}{\Gamma(1-\beta)}$$

for every $f \in \mathcal{C}^1$ vanishing at 0.

Hence $$\left(1 - \frac{\Gamma(1/q)}{\Gamma(1-\beta)\Gamma(\beta+1/q)}\right) N_p(D^\beta f) \leq N_p(h)$$

The coefficient in the left-hand member is positive, in view of lemma 28.

Then there exists a constant $c_{p,\alpha,\beta}$ such that

$$N_{\beta,p}(f)^p = N_p(D^\beta f)^p \leq c_{p,\alpha,\beta}\iint_{I^2}\frac{|f(x)-f(y)|^p\,dxdy}{|x-y|^{1+p\alpha}}$$

Now turn to the last inclusion. Write

$$\|I^\gamma f\|_{\delta,p}^p = 2\int_0^1 \frac{dt}{t^{1+p\delta}}\int_t^1 |I^\gamma f(x) - I^\gamma f(x-t)|^p\,dx$$

$$\left(\int_t^1 |I^\gamma f(x) - I^\gamma f(x-t)|^p\,dx\right)^{1/p} \leq \frac{N_p(f)}{\Gamma(\gamma)}\int_0^\infty |x^{\gamma-1} - (x-t)_+^{\gamma-1}|\,dx \leq c.N_p(f)t^\gamma$$

$$\|I^\gamma f\|_{\delta,p}^p \leq c' N_p(f)^p \int_0^1 \frac{t^{p\gamma}\,dt}{t^{1+p\delta}} \leq c'' N_p(f)^p$$

Choosing $\beta' \in ]\beta,\alpha[$ or $\gamma' \in ]\delta,\gamma[$ and applying proposition 1 gives compactness.

It remains to prove
**28 Lemma :** *if* $\alpha,\beta \in ]0,1[$ *and* $\alpha+\beta>1$, *then* $\Gamma(\beta) < \Gamma(\alpha+\beta)\Gamma(1-\alpha)$

Proof : one has $\alpha+\beta-1>0$ then $t^{\alpha+\beta-1}<1$, and hence

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 t^{\beta-1}(1-t)^{\alpha-1}dt < \int_0^1 t^{-\alpha}(1-t)^{\alpha-1}dt = \Gamma(\alpha)\Gamma(1-\alpha)$$

**29 Remark :** if $f$ is Borel and $\|f\|_{\alpha,p}$ is finite, one can prove the existence of a constant $c$ such that $f - c \in \mathcal{S}_{\alpha,p}$ (hence $c = f(0)$).

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