Some combinations of Asian, Parisian and Barrier Options

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1 Introduction

Amongst the large variety of path-dependent options, barrier options enjoy the feature of having been traded and discussed in the literature for quite some time. Back in 1973, Merton offered in his seminal paper a pricing formula for an option whose pay-off is restricted by a floor knock-out boundary and a few years later, Goldman, Sosin and Gatto (1979) provided closed form solutions for all types of single barrier options. Over the last few years, barrier options have become increasingly popular since they may produce, at a lower cost than standard options, the appropriate hedge in a number of risk management strategies (see Reiner and Rubinstein (1991)). Some of them combine several exotic features. For instance, it is well known that when implied volatilities are trading at historically high levels, going to Asian instruments reduces this volatility and the option price, as long as the risk-adjusted drift of the underlying asset is positive or within some interval (see Geman-Yor (1993) for a thorough discussion of this issue). As a way to cut premium even further, the user may choose an instrument which also has a knock-out feature; for instance, consider an American treasurer who is expecting a series of cash-flows denominated in deutschemarks and wants to buy Asian options DM/\$. If the current spot level is DM/\$ 1.4050, the premium can be reduced by the addition of a knock-out barrier at DM/\$ 1.4850. This Asian barrier put is still an attractive hedging tool since, if the barrier is reached, it means that the

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underlying asset is typically moving in favor of the cash position the option buyer wants to protect.

In the same manner, by buying Asian calls on oil, a major airline company may hedge its exposure against an increase in the cost of fuel (both because its needs in resupplying are regularly spread over time and because the nature of oil production, namely, the long time between extraction and delivery, entails that oil indices are typically arithmetic averages). The company may want to reduce the cost of its coverage by asking a down and out specification on the Asian call it will buy on the OTC markets.

French and Australian financial institutions have recently traded so-called “Parisian” options, whose pay-off is contingent on the fact that the underlying asset remains below or above a given value for a time period longer than a fixed number called the window (see Chesney, Jeanblanc-Picqué, Yor (1995) for a complete description and valuation of these instruments). For a window length equal to zero, the Parisian option reduces to a standard barrier option. When the window is extended until maturity, the Parisian option reduces to a standard European option. In the intermediate case, the option presents its “Parisian” feature and becomes a flexible financial tool which has some interesting properties : for instance, for some values of the parameters, when the underlying asset price is close to the barrier or when the size of the window is small, its value is a decreasing function of the volatility. Therefore, it allows traders to bet in a simple manner on a decrease of volatility. Last but not least, as far as down-and-out barrier options are concerned, an influential agent in the market who has written such options and sees the price approach the barrier may try to push the price further down, even momentarily and the cost of doing so may be smaller than the option payoff. In the the case of Parisian options, this would be more difficult and more expensive. Therefore, as in the case of Asian versus standard options, the possibility of market manipulations is reduced.

In the “Asian Parisian” case, the excursion condition is relative to the underlying asset, but the pay-off at maturity corresponds to an Asian option. The price of an “ Asian Parisian” option is, ceteris paribus, lower than the Asian option price. For instance, an up-and-in “Asian Parisian” option knocks in when the underlying price remains above a given level for a period of time at least equal to the window. In this case, the option only represents a hedge in the worst cases, namely when both the spot price and its average are high.

This paper addresses some of the valuation problems, in the Black and Scholes setting of a geometric Brownian motion for the underlying asset dynamics, for options whose pay-off is related to the terminal price of the stock and an arithmetic average of fixing and/or involves stopping times related to excursions. In all cases, we are able to provide at least the Laplace transform in time of the option price under a form whose complexity varies with the number of exotic features. A discussion of the organisation of this paper is postponed until the end of section 2. We emphasize that we do not give closed form formulas for the general case, but we aim to develop a methodology which may be used in many cases.

2 The setting

We assume in this paper that the underlying asset dynamics are driven by the equation

$$\begin{cases} dS_s &= S_s(\nu\, ds + dB_s) \\ S_0 &= 1\,, \end{cases}$$

where $(B_s, s \geq 0)$ is a standard Brownian motion under the usual risk-adjusted probability. We have set the volatility equal to 1 in order to obtain simpler formulas, the general drift ν allows for dividend-paying stocks and currencies. We denote by $B_s^{(\nu)} = B_s + \nu s$ the Brownian motion with drift ν. In previous research, we have been interested in options whose prices are, up to the discount factor, defined by the following quantities (t stands for the maturity)

- Asian options :

$$E\left[(A_t^{(\nu)} - k)^+\right]$$

  where $A_t^{(\nu)} = \displaystyle\int_0^t ds\, \exp(2B_s^{(\nu)})$. Asian options are studied in Geman-Yor (1993). See also Kemna and Vorst (1990) and Rogers and Shi (1995) for different approaches.

- Parisian options : Recall the definition of an up-and-out "Parisian" option: the owner of this option loses it if the underlying asset price $(S_s, s \geq 0)$ reaches a level L before maturity t and remains constantly *above* this level for a time interval longer than a fixed number c, called the option window. If not, the owner will receive the pay-off $(S_t - k)^+$. Therefore, we consider excursions[1] of the process $(B_s^{(\nu)}, s \geq 0)$ above a given barrier. In this paper, we assume that the level L is equal to the spot price at initial time, which implies that the excursions for the process $B_s^{(\nu)}$ are at (more correctly : away from) the level 0. The general case can easily be derived by waiting until the first hitting time of the barrier and then translating that level to 0. See Chesney et al. (1995) for more details.

  Let g_s be the left extremity of the excursion which straddles time s, and

$$H_c^+ = \inf\{s : \mathbb{1}_{B_s^{(\nu)}>0}(s - g_s) \geq c\}\,,$$

  the first time when an excursion above 0 is "older" than the window c. The price of an up-and-out Parisian call is given by

$$E\left[\mathbb{1}_{H_c^+>t}\left(\exp(B_t^{(\nu)}) - k\right)^+\right].$$

[1] See the Appendix for a precise definition.

Our methodology applies also to other Parisian options, namely down and knock-in options.

- Double barrier options :

$$E\left[\mathbb{1}_{T_{-a,b}>t}\left(\exp(B_t^{(\nu)}) - k\right)^+\right],$$

where $T_{-a,b} = \inf\{s : B_s^{(\nu)} \notin [-a, b]\}$. Such options are studied by a number of authors, including Kunimoto and Ikeda (1992), He, Keirstead and Rebholtz (1995) and Geman and Yor (1995).

This has led us to consider the mixed general quantities

$$\begin{aligned}\sigma_+^{(\nu)}(a,b,k;t) &\stackrel{def}{=} E\left[\mathbb{1}_{\Sigma>t}\left(a\exp(B_t^{(\nu)}) + bA_t^{(\nu)} - k\right)^+\right] \\ \sigma_-^{(\nu)}(a,b,k;t) &\stackrel{def}{=} E\left[\mathbb{1}_{\Sigma\leq t}\left(a\exp(B_t^{(\nu)}) + bA_t^{(\nu)} - k\right)^+\right],\end{aligned} \tag{1}$$

where Σ is a stopping time in the filtration generated by $(B_t^{(\nu)})$. We develop below the computations related to the particular cases $\Sigma = H_{c,d}$ and $\Sigma = T_{-a,b}$ where

$$H_{c,d} \stackrel{def}{=} \inf\{s : \mathbb{1}_{B_s^{(\nu)}>0}(s - g_s) \geq c \quad \text{or} \quad \mathbb{1}_{B_s^{(\nu)}<0}(s - g_s) \geq d\}\,.$$

We use the concise notation

$$\begin{aligned}H_c^+ &\stackrel{def}{=} H_{c,\infty} = \inf\{s : \mathbb{1}_{B_s^{(\nu)}>0}(s - g_s) \geq c\} \\ H_d^- &\stackrel{def}{=} H_{\infty,d}\,.\end{aligned}$$

In the case $\Sigma = H_c^+$ (resp. H_d^-), only excursions above 0 (resp. below 0) are relevant. To simplify the presentation, we also introduce

$$H_c = \inf\{s : s - g_s \geq c\}\,,$$

since this stopping time allows us to develop our methodology, without taking care of the signs of excursions.

The case $a = 0, \Sigma = t$ is the Asian option case; $b = 0, \Sigma = H_c^+, \sigma_+^{(\nu)}$ (resp. $b = 0, \Sigma = H_c^+, \sigma_-^{(\nu)}$) is the up-and-out (resp. up-and-in) Parisian option case; $b = 0, \Sigma = T_{-a,b}$ is the double barrier case.

The reader will note that the computation of $\sigma_+^{(\nu)}(a,b,k;t) + \sigma_-^{(\nu)}(a,b,k;t)$ reduces to that of $E\left[\left(a\exp(B_t^{(\nu)}) + bA_t^{(\nu)} - k\right)^+\right]$. This last expression involves the law of the pair $(B_t^{(\nu)}, A_t^{(\nu)})$ and is computed in section 5.

Therefore, we restrict our attention to $\sigma^{(\nu)}(a,b,k;t) \stackrel{def}{=} \sigma_-^{(\nu)}(a,b,k;t)$. We derive in this paper the expression of the Laplace transform of $\sigma^{(\nu)}$ with respect to time. This method could even be extended to the situation where the pay-off is a function of $B_t^{(\nu)}$ and $A_t^{(\nu)}$. In the particular case $a=b=0$, the Laplace transform of Σ is also obtained, hence the price of the corresponding boost option[2].
In fact, equality (1) covers several types of contingent claims which may be useful in specific situations. For example, our method yields the price of Asian floating options, i.e., with pay-off $(A_t^{(\nu)} - \exp B_t^{(\nu)})^+$, which corresponds to the case $k=0, b=1, a=-1, \Sigma=t$ and the price of Asian spread options, with pay-off the positive part of the difference between two independent Asian pay-offs.

The paper is organised as follows: in the next section, we show that, using a Laplace transform in time, the problem can be split into two subproblems. The first one reduces to the computation of the joint law of $(\Sigma, \exp B_\Sigma^{(\nu)}, A_\Sigma^{(\nu)})$ and is studied in section 4. The second one is to find the law of the pair $(B_t^{(\nu)}, A_t^{(\nu)})$ and is solved in section 5. An Appendix gathers the main definitions and results about the different stochastic processes which are used in this paper.

3 Laplace Transform in time

As observed in previous works by the authors, the Markov property and time changes can be used in an essential way for such computations. In what follows, $\overline{B}^{(\nu)}$ denotes a Brownian motion with drift ν, assumed to be independent of the original Brownian motion $(B_s, s \geq 0)$. We want to compute the time Laplace transform of $\sigma^{(\nu)}$ which, thanks to the strong Markov property applied at time Σ, can be written as

$$\int_0^\infty dt\, e^{-\lambda t} \sigma^{(\nu)}(a,b,k;t)$$

$$= E\left[\exp(-\lambda\Sigma) \int_0^\infty dt\, e^{-\lambda t}\left(a \exp(B_\Sigma^{(\nu)} + \overline{B}_t^{(\nu)}) + b\,[A_\Sigma^{(\nu)} + \overline{A}_t^{(\nu)} \exp(2B_\Sigma^{(\nu)})] - k\right)^+\right].$$

Thus, clearly our original problem decomposes into two subproblems:

- Finding the joint law of $(\Sigma, \exp B_\Sigma^{(\nu)}, A_\Sigma^{(\nu)})$. We call this problem (JL), which itself decomposes into $\begin{cases} (JL)^1, \text{ for } \Sigma = H_c \text{ or } \Sigma = H_{c,d} \\ (JL)^2, \text{ for } \Sigma = T_{-a,b}\,, \end{cases}$

- Computing the resolvent type quantity

$$v^{(\nu)}(\lambda, \overline{a}, \overline{b}, \overline{k}) \stackrel{def}{=} E\left[\int_0^\infty dt\, e^{-\lambda t}\left(\overline{a} \exp \overline{B}_t^{(\nu)} + \overline{b}\, \overline{A}_t^{(\nu)} - \overline{k}\right)^+\right].$$

  We use bars to avoid confusion with the original parameters $a, b, k, \ldots$. We refer to this second problem as problem (V). The particular case $\bar{a} = 0$

[2] Recall that, for a boost option, the pay-off is proportional to the time spent in a band.

leads to the Asian option price, whereas the case $\bar{a} = -\bar{b}$ leads to the Asian floating options price.

4 A solution of problem (JL)

4.1 On problem $(JL)^1$

Here the problem is to compute the law of the triple

$$(H_c, \exp B^{(\nu)}_{H_c},\, A^{(\nu)}_{H_c})\,.$$

The more general case when $\Sigma = H_{c,d}$ will be studied at the end of this section.

The Cameron-Martin formula allows one to reduce the problem to the case $\nu = 0$. We recall that we are interested in excursions of the Brownian motion away from 0 and that

$$g_t = \sup\{s \le t, B_s = 0\}\ , \quad H_c = \inf\{t : t - g_t \ge c\}\,.$$

For this purpose, we use the following[3]

Proposition 1
(i) The σ-field $\mathcal{F}^-_{g_{H_c}}$, the random variable $\eta_c \stackrel{def}{=} sgn\,(B_{H_c})$, and the process $\{|B_{g_{H_c}} + u|\,, u \le c\}$ are independent; furthermore, η_c is a symmetric random Bernoulli variable, and $\{|B_{g_{H_c}} + u|\,, u \le c\}$ is a Brownian meander with length c.
(ii) For any $\mathbb{R}_+$-valued, $(\mathcal{F}_t)$-predictable process $(z_t, t \ge 0)$, the following relationship holds

$$\sqrt{\frac{\pi c}{2}}\, E[z_{g_{H_c}}] = \int_0^\infty ds\, E[z_{\tau_s} \mathbb{1}_{\Delta(\tau_s) \le c}]\,,$$

where $\tau_s = \inf\{u : \ell_u > s\}$, ℓ is the local time of (B_t) at 0, and

$$\Delta(\tau_s) = \sup_{u \le s}(\tau_u - \tau_{u^-})$$

is the maximum length of excursions up to time τ_s.

Proof : The first part relies on standard properties of the Brownian meander which are recalled in the Appendix. The second part follows from the "balayage" formula (see Revuz-Yor (1994), chap. VI, sect. 4) which states that for any bounded, $(\mathcal{F}_t)$-predictable process $(z_t, t \ge 0)$, the process

$$z_{g_t}|B_t| - \int_0^t d\ell_u\, z_u$$

[3] See the Appendix for the corresponding definitions.

is a $(\mathcal{F}_t)$-martingale; hence, by projection on $(\mathcal{F}^-_{g_t})$, the process

$$z_{g_t}\,|\mu_t| - \int_0^t d\ell_u\, z_u$$

is a $(\mathcal{F}^-_{g_t})$-martingale, where $\mu_t \stackrel{def}{=} \mathrm{sgn}\,(B_t)\sqrt{\frac{\pi}{2}(t-g_t)}$ is the so-called Azéma martingale; recall that μ_t and $|\mu_t|-\ell_t$ are $(\mathcal{F}^+_{g_t})$-martingales. Then, we use the stopping time theorem, at time H_c, to obtain

$$\sqrt{\frac{\pi c}{2}}\, E(z_{g_{H_c}}) = E(\int_0^{H_c} d\ell_u\, z_u)\,.$$

The final formula follows by making the obvious time change in the integral with respect to $d\ell_u$. □

We now explain how to exploit the above results to proceed with the solution of $(JL)^1$.

a) Our aim is to compute the law of the triple $(H_c,\, B_{H_c},\, A_{H_c})$. This problem is solved via the joint Laplace transform of this law, i.e.,

$$u(\alpha,k,\theta;c) \stackrel{def}{=} E\Big[\exp(\alpha B_{H_c} - \frac{k^2}{2}H_c - \frac{\theta^2}{2}A_{H_c})\Big]$$

which, from Proposition 1 (i), is equal to :

$$u(\alpha,k,\theta;c) = u_b(k,\theta;c)\,\exp(-\frac{k^2 c}{2})\,u_m(\alpha,\theta;c)\,,$$

where

$$u_b(k,\theta;c) \stackrel{def}{=} E\Big[\exp\Big(-\frac{1}{2}(k^2 g_{H_c} + \theta^2 A_{g_{H_c}})\Big)\Big]$$

and

$$u_m(\alpha,\theta;c) \stackrel{def}{=} E\Big[\exp(\alpha B_{H_c} - \frac{\theta^2}{2}\int_{g_{H_c}}^{H_c} du\, \exp(2B_u))\Big]\,.$$

We now explain the mnemonic for u_b and u_m.
i) We remark that the definition of u_b involves the trajectories of the Brownian motion between 0 and g_{H_c}, and that the law of the process $(\frac{1}{g_{H_c}}B_{u g_{H_c}}\,;u\le 1)$ is equivalent (i.e., mutually absolutely continuous with respect) to that of the Brownian bridge which may be represented as $(\frac{1}{g_t}B_{u g_t}\,;u\le 1)$; however, we shall not use, nor prove, this absolute continuity result here.
ii) The definition of u_m involves the trajectories of the Brownian meander between g_{H_c} and H_c and the process $(|B_{g_{H_c}}+u|\,,u\le c)$ is a Brownian meander with length c.

Thus, again, problem $(JL)^1$ may be split into two subproblems, which we shall denote by $(JL)^1_m$ and $(JL)^1_b$.

b) In order to solve problem $(JL)^1_b$, we use the multiplicative "master formula" of excursion theory (See Revuz-Yor, chap. XII) which implies that, for any Borel function $f : \mathbb{R} \to \mathbb{R}_+$,

$$E[\mathbb{1}_{\Delta(\tau_s)\leq c}\, \exp(-A^f_{\tau_s})] = \exp(-sJ(c,f)\,)\,,$$

where $A^f_t = \int_0^t du\, f(B_u)$ and[4]

$$J(c,f) \overset{def}{=} \int \mathbf{n}(d\epsilon)\Big(1 - \mathbb{1}_{V(\epsilon)\leq c} \exp(-A^f_V(\epsilon))\Big)\,. \tag{2}$$

As a consequence of Proposition 1 (ii), one obtains the important result :

Corollary 1 *Using the previous notation, for any positive measurable function f*

$$\sqrt{\frac{\pi c}{2}}\, E[\exp(-A^f_{g_{H_c}})] = 1/J(c,f)\,. \tag{3}$$

4.1.1 On problem $(JL)^1_m$

This consists in the computation of $u_m(\alpha,\theta;c)$ which can be expressed in terms of the Brownian meander. Indeed,

$$\begin{aligned} u_m(\alpha,\theta;c) &= \frac{1}{2}\, E\Big[\exp\Big(\alpha\sqrt{c}\, m_1 - \frac{\theta^2 c}{2}\int_0^1 du\, \exp(2\sqrt{c}\, m_u)\Big)\Big] \\ &+ \frac{1}{2}\, E\Big[\exp\Big(-\alpha\sqrt{c}\, m_1 - \frac{\theta^2 c}{2}\int_0^1 du\, \exp(-2\sqrt{c}\, m_u)\Big)\Big]\,, \end{aligned}$$

where now $(m_u, u\leq 1)$ denotes the standard Brownian meander (i.e., with length 1). We denote by M^c the law of the Brownian meander $(m^{(c)}_u, u\leq c)$ with length c, i.e.,

$$(m^{(c)}_u = \sqrt{c}\, m_{u/c}, u \leq c)$$

on the canonical space $(C(\mathbb{R}_+,\mathbb{R}_+),\mathcal{R}_\infty)$.
Remark that, in the particular case $\theta = 0$ ("Parisian" case), the computation of $u_m(\alpha,0;c)$ follows from the expression of the law of m_1 :

$$P(m_1 \in dx) = x\exp(-x^2/2)\mathbb{1}_{x>0}\, dx\,.$$

To compute $u_m(\alpha,\theta;c)$, we proceed by looking at the Laplace transform in the variable $\frac{k^2}{2}$ of $c \to \frac{u_m(\alpha,\theta;c)}{\sqrt{2\pi c}}$ ([5]); thus, we search for an expression of

$$\Phi_\pm(k,\alpha,\theta) \overset{def}{=} \int_0^\infty \frac{dc\, \exp(-k^2c/2)}{\sqrt{2\pi c}}\, M^c\Big(\exp\Big[\pm\alpha R_c - \frac{\theta^2}{2}\int_0^c du\, \exp(\pm 2R_u)\Big]\Big)\,,$$

[4]See the Appendix for the definitions of the Itô measure $\mathbf{n}$ and of the lifetime V.

[5]Dividing by $\sqrt{2\pi c}$ simplifies subsequent computations; this will be clear in the next lines.

where $(R_u, u \geq 0)$ is the canonical process. We know from excursion theory (see, e.g., Revuz-Yor, Exercise 4.18, chap. XII) that

$$\begin{aligned}\Phi_\pm(k,\alpha,\theta) &= \int_0^\infty da\, E_a\Big[\exp -\Big(\frac{k^2}{2}T_0 \mp \alpha a + \frac{\theta^2}{2}\int_0^{T_0} du\, \exp\left(\pm 2B_u\right)\Big)\Big] \\ &= \int_0^\infty da\, \exp(\pm\alpha a)\, E_{\pm a}\Big[\exp\Big(-\frac{1}{2}(k^2 T_0 + \theta^2 \int_0^{T_0} du \exp\left(2B_u\right))\Big)\Big]\,,\end{aligned}$$

where T_0 denotes the first time Brownian motion reaches 0 and E_a denotes the expectation under the law of Brownian motion starting at a. Note that the integral with $\exp(a\alpha)$ is finite for $\alpha < k$.

The crucial point in our solution of $(JL)^1_m$ is the following

Proposition 2 *Let $f : I\!R \to I\!R_+$ be a locally bounded function. Then the function*

$$u(a) \equiv u^f(k,a) \stackrel{def}{=} E_a\Big[\exp -\Big(\frac{k^2}{2}T_0 + \int_0^{T_0} du\, f(B_u)\Big)\Big] \tag{4}$$

is the unique bounded solution of the Sturm-Liouville equation

$$\frac{1}{2}\,u'' = \Big(\frac{k^2}{2} + f\Big)\,u\,; \qquad u(0) = 1\,.$$

As particular examples, one has, for $a \geq 0$, $k \geq 0$, $\theta > 0$:

$$E_a\Big[\exp -\tfrac{1}{2}(k^2 T_0 + \theta^2 A_{T_0})\Big] = \frac{K_k(\theta e^a)}{K_k(\theta)} \tag{5$_+$}$$

$$E_{-a}\Big[\exp -\tfrac{1}{2}(k^2 T_0 + \theta^2 A_{T_0})\Big] = \frac{I_k(\theta e^{-a})}{I_k(\theta)}\,, \tag{5$_-$}$$

where I_k and K_k are modified Bessel functions.

Notation 1. In what follows, the two formulas $(5)_\pm$ will play an important role; related formulas (f) involving the positive (resp. negative) level a (resp : $-a$) and/or the function $\exp(2x)$ (resp. $\exp(-2x)$) will be presented as formulas $(f)_+$ (resp $(f)_-$).
2. We denote by $P^{(\nu)}_\alpha$ (or, when more convenient, ${}^{(d)}P_\alpha$) the law of the Bessel process[6] $(R^{(\nu)}_u, u \geq 0)$ of index ν (of dimension d), starting at α.

Proof of Proposition 2 : The general statement follows from the optional sampling theorem, Itô's formula, and/or the Feynman-Kac formula. Rather than deducing formula $(5)_\pm$ directly from the general case, we shall connect these formulas with computations done in Pitman-Yor (1981).

[6]See the Appendix for some definitions.

We denote by $P^{(0)}_\alpha$ the law of the 2-dimensional Bessel process $(R^{(0)}_u, u \geq 0)$, starting from $\alpha \geq 0$. We have exhibited in previous research (Geman-Yor (1993)) the power of Lamperti's representation of a geometric Brownian Motion as a time-changed Bessel process (see Revuz-Yor, chap. XI)

$$\exp(B_t + \nu t) = R^{(\nu)}_{A^{(\nu)}_t} . \tag{6}$$

We also introduce $C^{(\nu)}_u = \displaystyle\int_0^u \frac{ds}{(R^{(\nu)}_s)^2}$. Let us remark that $C^{(\nu)}$ and $A^{(\nu)}$ are inverses of each other, namely

$$C^{(\nu)}_u = \inf\{t \mid A^{(\nu)}_t > u\} \tag{7}$$

and $R^{(\nu)}_u = \exp(B_{C^{(\nu)}_u} + \nu C^{(\nu)}_u)$. Then, using the representation $\exp B_t = R_{A^{(0)}_t}$, and denoting by T_1 the hitting time $T_1 = \inf\{t : R_t = 1\}$, we deduce that the left-hand side of $(5)_+$ is equal to

$$E^{(0)}_{e^a}\left(\exp -\frac{1}{2}\left(k^2 \int_0^{T_1} \frac{du}{R^2_u} + \theta^2 T_1\right)\right) = \frac{K_k(\theta e^a)}{K_k(\theta)} , \tag{8}$$

the last equality being borrowed from Pitman-Yor (1981) (Proposition 2.3). The same argument leads to formula $(5)_-$. □

Warning : In Yor (1993), points c) and d) of lemma 1, the right-hand sides of $(5)_+$ and $(5)_-$ have erroneously been inverted. However, this does not affect the subsequent results in Yor (1993).

To proceed further, we shall use the Hartman distributions, the definition of which we now present. Let $0 < r < R < \infty$. P. Hartman (1976) showed from a purely analytical viewpoint, the existence of two positive integrable functions $h^{\uparrow}_{r,R}$ and $h^{\downarrow}_{r,R}$ such that

$$\frac{I_k(r)}{I_k(R)} = \int_0^\infty \exp(-\frac{k^2}{2}t)\, h^{\uparrow}_{r,R}(t)\, dt$$

and

$$\frac{K_k(R)}{K_k(r)} = \int_0^\infty \exp(-\frac{k^2}{2}t)\, h^{\downarrow}_{r,R}(t)\, dt .$$

Now, using Proposition 2 together with the definition of the Hartman densities $h^{\uparrow}_{r,R}$ and $h^{\downarrow}_{r,R}$, we obtain

$$\begin{aligned}
\Phi_+(k, \alpha, \theta) &= \int_0^\infty da\, \exp(\alpha a)\, \frac{K_k(\theta e^a)}{K_k(\theta)} \\
&= \int_0^\infty da\, \exp(\alpha a) \int_0^\infty dc\, \exp(-\frac{k^2 c}{2})\, h^{\downarrow}_{\theta,\theta e^a}(c)
\end{aligned}$$

and

$$\begin{aligned}\Phi_-(k,\alpha,\theta) &= \int_0^\infty da\ \exp(-\alpha a)\,\frac{I_k(\theta e^{-a})}{I_k(\theta)}\\ &= \int_0^\infty da\ \exp(-\alpha a)\int_0^\infty dc\ \exp(-\frac{k^2 c}{2})\,h^{\uparrow}_{\theta e^{-a},\theta}(c)\,.\end{aligned}$$

Next, recalling that $\Phi_\pm(k,\alpha,\theta)$ is a Laplace transform with respect to $k^2/2$, which involves $(M^c, c>0)$, we obtain

$$\frac{1}{\sqrt{2\pi c}}\,M^c\Big(\exp\Big[-\alpha R_c-\frac{\theta^2}{2}\int_0^c du\ \exp(-2R_u)\Big]\Big)=\int_0^\infty da\,e^{-\alpha a}\,h^{\uparrow}_{\theta e^{-a},\theta}(c)$$

and

$$\frac{1}{\sqrt{2\pi c}}\,M^c\Big(\exp\Big[\alpha R_c-\frac{\theta^2}{2}\int_0^c du\ \exp(2R_u)\Big]\Big)=\int_0^\infty da\,e^{\alpha a}\,h^{\downarrow}_{\theta,\theta e^{a}}(c)\,.$$

These formulas yield a complete solution of problem $(JL)^1_m$:

Proposition 3 *The function $u_m(\alpha,\theta;c)$ is given by*

$$u_m(\alpha,\theta;c)=\sqrt{\frac{\pi c}{2}}\ \int_0^\infty da\,(e^{-\alpha a}\,h^{\uparrow}_{\theta e^{-a},\theta}(c)+e^{\alpha a}\,h^{\downarrow}_{\theta,\theta e^{a}}(c))\,.$$

where the Hartman densities $h^{\uparrow}_{r,R}$ and $h^{\downarrow}_{r,R}$ are defined, via Laplace transform, by

$$\frac{I_k(r)}{I_k(R)}=\int_0^\infty \exp(-\frac{k^2}{2}t)\,h^{\uparrow}_{r,R}(t)\,dt$$

and

$$\frac{K_k(R)}{K_k(r)}=\int_0^\infty \exp(-\frac{k^2}{2}t)\,h^{\downarrow}_{r,R}(t)\,dt\,.$$

These formulas are still valid in the case $\theta=0$, using equivalent expressions for the Bessel functions (See the Appendix).

4.1.2 More results about Hartman densities

In order to understand better the Hartman densities, we desintegrate formula (4) with respect to the law of T_0.

Proposition 4 *Using the notation introduced in Proposition 2, we have*

$$u^f(k,a)=\int_0^\infty dt\ \exp(-\frac{k^2 t}{2})\,H_f(t,a)\,,$$

where

$$H_f(t,a) = \frac{a}{\sqrt{2\pi t^3}}\ \exp(-\frac{a^2}{2t})\,{}^{(3)}E_0\Big[\exp - \int_0^t du\, f(R_u)\Big| R_t = a\Big] \tag{9}$$

and ${}^{(3)}P_0$ *denotes the law of the 3-dimensional Bessel process starting from 0. As particular examples, one has, for any* $a > 0$*,*

$$\frac{a}{\sqrt{2\pi c^3}}\exp(-\frac{a^2}{2c})\ {}^{(3)}E_0\Big[\exp\Big(-\frac{\theta^2}{2}\int_0^c du\, \exp(-2R_u)\Big)\ \Big|\ R_c = a\Big] = h^{\uparrow}_{\theta e^{-a},\theta}(c) \quad (10)_-$$

$$\frac{a}{\sqrt{2\pi c^3}}\exp(-\frac{a^2}{2c})\ {}^{(3)}E_0\Big[\exp\Big(-\frac{\theta^2}{2}\int_0^c du\, \exp(2R_u)\Big)\Big|\ R_c = a\Big] = h^{\downarrow}_{\theta,\theta e^{a}}(c)\ . \quad (10)_+$$

Proof : The right-hand side of (4) may be written as

$$\int_0^\infty dt \frac{a}{\sqrt{2\pi t^3}}\ \exp(-\frac{a^2}{2t})\ \exp(-\frac{k^2 t}{2})\ E_a[\exp - \int_0^{T_0} du\, f(B_u))|T_0 = t]\,.$$

Furthermore, using Williams' time reversal and conditioning with respect to $L_a = \sup\{t : R_t = a\}$, we obtain

$$E_a\Big[\exp - \int_0^{T_0} du\, f(B_u)\Big| T_0 = t\Big] = {}^{(3)}E_0\Big[\exp - \int_0^t du\, f(R_u)\Big|\ R_t = a\Big]\,. \qquad \square$$

Now, we investigate limits in formulas (9) and $(10)_\pm$ as $a \to 0$. In particular, we find that

$$h^{\uparrow}_{\theta}(c) \stackrel{def}{=} \lim_{a\to 0^+} \frac{1}{a} h^{\uparrow}_{\theta e^{-a},\theta}(c) \quad \text{and} \quad h^{\downarrow}_{\theta}(c) \stackrel{def}{=} \lim_{a\to 0^+} \frac{1}{a} h^{\downarrow}_{\theta,\theta e^{a}}(c)$$

exist and satisfy

$$h^{\uparrow}_{\theta}(c) = \frac{1}{\sqrt{2\pi c^3}}\ {}^{(3)}E_0\Big[\exp\Big(-\frac{\theta^2}{2}\int_0^c du\, \exp(-2R_u)\Big)\Big| R_c = 0\Big] \quad (11)_-$$

and

$$h^{\downarrow}_{\theta}(c) = \frac{1}{\sqrt{2\pi c^3}}\ {}^{(3)}E_0\Big[\exp\Big(-\frac{\theta^2}{2}\int_0^c du\, \exp(2R_u)\Big)\Big| R_c = 0\Big]\,. \quad (11)_+$$

More generally, we obtain the following

Proposition 5 *Using the notation of Propositions 2 and 4 , one obtains the existence of* $K_f(t) \stackrel{def}{=} \lim_{a\to 0^+} \dfrac{H_f(t,a)}{a}$ *and the equality*

$$K_f(t) = \frac{1}{\sqrt{2\pi t^3}}\ {}^{(3)}E_0\Big[\exp - \int_0^t du\, f(R_u)\Big| R_t = 0\Big]\,.$$

Furthermore, this function K_f *is characterized via the following Laplace transform:*

$$\begin{aligned} -\frac{\partial}{\partial a}\Big|_{a=0^+} u^f(k,a) &= k + \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\exp(-\frac{k^2 t}{2})\Big(1 - \sqrt{2\pi t^3}\, K_f(t)\Big) \\ &= \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\Big(1 - \exp(-\frac{k^2 t}{2})\ {}^{(3)}E_0\Big[\exp - \int_0^t ds\, f(R_s)\Big| R_t = 0\Big]\Big)\,. \end{aligned}$$

In particular, one obtains that

$$\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\Big(1-\exp(-\frac{k^2t}{2})\ {}^{(3)}E_0\Big[\exp\Big(-\frac{\theta^2}{2}\int_0^t ds\ \exp(-2R_s)\Big)\Big|R_t=0\Big]\Big)$$

is equal to

$$\frac{\theta K_{k+1}(\theta)}{K_k(\theta)}-k=\frac{\theta K_{k-1}(\theta)}{K_k(\theta)}+k\,, \qquad (12)_-$$

and

$$\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\Big(1-\exp(-\frac{k^2t}{2})\ {}^{(3)}E_0\Big[\exp\Big(-\frac{\theta^2}{2}\int_0^t ds\ \exp(2R_s)\Big)\Big|R_t=0\Big]\Big)$$

is equal to

$$\frac{\theta I_{k-1}(\theta)}{I_k(\theta)}-k=\frac{\theta I_{k+1}(\theta)}{I_k(\theta)}+k\,. \qquad (12)_+$$

Proof:
1) We divide by a the two sides of the equality

$$u^f(k,a)-1=(u^f(k,a)-e^{-ka})+(e^{-ka}-1)\,.$$

Then, using

$$e^{-ka}=\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\ a\exp[-\frac{1}{2}(k^2t+\frac{a^2}{t})]\,,$$

we obtain

$$-\frac{\partial}{\partial a}\Big|_{a=0^+} u^f(k,a)=k+\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\exp(-\frac{k^2t}{2})\Big(1-\ {}^{(3)}E_0\Big[\exp(-\int_0^t ds\, f(R_s))\Big|R_t=0\Big]\Big)\,.$$

We also note that $k=\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\Big(1-\exp(-\frac{k^2t}{2})\Big)$, which leads to the second form of the derivative of $u^f(k,a)$.
2) To obtain $(12)_\pm$, we apply the previous result together with the recurrence relations between Bessel functions and their derivatives (see Lebedev (1972), p. 110)

$$\begin{cases} \dfrac{\theta K_{k+1}(\theta)}{K_k(\theta)}-k &= \dfrac{\theta K_{k-1}(\theta)}{K_k(\theta)}+k \\ \dfrac{\theta I_{k-1}(\theta)}{I_k(\theta)}-k &= \dfrac{\theta I_{k+1}(\theta)}{I_k(\theta)}+k \end{cases}$$

and

$$\begin{cases} K_{k+1}(\theta) &= -K_k'(\theta)+\dfrac{k}{\theta}K_k(\theta) \\ I_{k-1}(\theta) &= I_k'(\theta)+\dfrac{k}{\theta}I_k(\theta) \end{cases}.$$

In particular, we deduce from formulas $(11)_\pm$ and $(12)_\pm$ some analytic representation of $h_\theta^\uparrow$ and $h_\theta^\downarrow$:

Corollary 2 *One has for $\theta, k > 0$*

$$\begin{aligned}1+\frac{\partial}{\partial k}\Big(\theta\frac{I_{k+1}(\theta)}{I_k(\theta)}\Big) &= k\int_0^\infty dt\exp(-\frac{k^2t}{2})\,t\,h_\theta^{\downarrow}(t)\\ 1+\frac{\partial}{\partial k}\Big(\theta\frac{K_{k-1}(\theta)}{K_k(\theta)}\Big) &= k\int_0^\infty dt\exp(-\frac{k^2t}{2})\,t\,h_\theta^{\uparrow}(t)\,.\end{aligned}$$

4.1.3 On problem $(JL)_b^1$

We are now interested in the computation of $u_b(k,\theta;c)$, which, from formula (3), amounts to the computation of

$$J_\pm(c,m)\stackrel{def}{=}\int \mathbf{n}_\pm(d\epsilon)\Big(1-\mathbb{1}_{V(\epsilon)\le c}\exp(-\frac{m^2}{2}\int_0^{V(\epsilon)}du\,\exp(2\epsilon_u)\,)\Big)\,,$$

where $\mathbf{n}_\pm$ is the Itô measure of positive (negative) excursions (details of this reduction will be given below).

Proposition 6 *The following formulas hold, for $\mu > 0$ and $m > 0$*

$$\int_0^\infty dc\,\exp(-\frac{\mu^2c}{2})\,J_+(c,m) = \Big(\frac{mK_{\mu+1}(m)}{K_\mu(m)}-\mu\Big)\frac{1}{\mu^2} \qquad (13)_+$$

$$\int_0^\infty dc\,\exp(-\frac{\mu^2c}{2})\,J_-(c,m) = \Big(\frac{mI_{\mu-1}(m)}{I_\mu(m)}-\mu\Big)\frac{1}{\mu^2}\,. \qquad (13)_-$$

Proof: We first remark that using Fubini's theorem, we obtain

$$\frac{\mu^2}{2}\int_0^\infty dc\,\exp(-\frac{\mu^2c}{2})\,J_\pm(c,m)$$

$$=\int \mathbf{n}_\pm(d\epsilon)(1-\exp(-\frac{m^2}{2}\int_0^{V(\epsilon)}du\exp(2\epsilon_u)\;-\frac{\mu^2}{2}V(\epsilon)))\,. \qquad (14)_\pm$$

On the other hand, we use formulas a) and b) from Lemma 1 of Yor (1993)

$$\begin{aligned}E[\exp-\frac{1}{2}(\mu^2\tau_s^+ + m^2A^+_{\tau_s})] &= \exp-\frac{s}{2}(\frac{mK_{\mu+1}(m)}{K_\mu(m)}-\mu)\\ E[\exp-\frac{1}{2}(\mu^2\tau_s^- + m^2A^-_{\tau_s})] &= \exp-\frac{s}{2}(\frac{mI_{\mu-1}(m)}{I_\mu(m)}-\mu)\,,\end{aligned}$$

where

$$A_t^\pm=\int_0^t ds\exp(2B_s)\,\mathbb{1}_{B_s\in\mathbb{R}^\pm}\quad,\quad \tau_s^\pm=\int_0^{\tau_s}du\,\mathbb{1}_{B_u\in\mathbb{R}^\pm}\,.$$

Then, the master multiplicative formula implies that the right-hand sides of $(14)_\pm$ are respectively equal to

$$\frac{1}{2}(\frac{mK_{\mu+1}(m)}{K_\mu(m)}-\mu)\,,\;\frac{1}{2}(\frac{mI_{\mu-1}(m)}{I_\mu(m)}-\mu)\,.$$

This proves, in particular, formulas $(13)_\pm$. $\square$

We now explain how the computation of $u_b(k,\theta;c)$ may be reduced to that of $J_\pm(c,m)$. Indeed, from formula (3), one has $u_b(k,\theta;c) = 1/J(c;k,\theta)$ where

$$J(c;k,\theta) = \int \mathbf{n}(d\epsilon)\Big(1 - 1\!\!1_{V(\epsilon)\le c} \exp[-\frac{1}{2}(\theta^2 \int_0^{V(\epsilon)} du \exp(2\epsilon_u) \; + k^2 V(\epsilon))]\Big)\,,$$

then, we write

$$\exp(-\frac{k^2 V}{2}) = \frac{k^2}{2}\int_V^\infty dx \exp(-\frac{xk^2}{2})$$

which, plugged into the previous formula, yields

$$J(c;k,\theta) =$$

$$\frac{k^2}{2}\int_0^\infty dx \exp(-\frac{xk^2}{2}) \int \mathbf{n}(d\epsilon)\left(1 - 1\!\!1_{V(\epsilon)\le c}\, 1\!\!1_{V(\epsilon)\le x} \exp[-\frac{\theta^2}{2}\int_0^{V(\epsilon)} du \exp(2\epsilon_u)]\right).$$

Therefore, $J(c;k,\theta)$ may be written in terms of $J_\pm(c,\theta)$

$$J(c;k,\theta) = \frac{k^2}{2}\int_0^\infty dx \exp(-\frac{xk^2}{2})\Big[J_+(c\wedge x,\theta) + J_-(c\wedge x,\theta)\Big]\,.$$

Therefore, the $(JL)_b^1$ problem is solved:

Proposition 7 *The function $u_b(k,\theta;c)$ is defined by*

$$u_b(k,\theta;c) = \left[\frac{k^2}{2}\int_0^\infty dx \exp(-\frac{xk^2}{2})\Big[J_+(c\wedge x,\theta) + J_-(c\wedge x,\theta)\Big]\right]^{-1},$$

where $J_\pm(c,\theta)$ are given, via Laplace transform by

$$\begin{aligned}
\int_0^\infty dc \, \exp(-\frac{\mu^2 c}{2})\, J_+(c,m) &= \Big(\frac{mK_{\mu+1}(m)}{K_\mu(m)} - \mu\Big)\frac{1}{\mu^2} \\
\int_0^\infty dc \, \exp(-\frac{\mu^2 c}{2})\, J_-(c,m) &= \Big(\frac{mI_{\mu-1}(m)}{I_\mu(m)} - \mu\Big)\frac{1}{\mu^2}\,.
\end{aligned}$$

Let us remark that the particular case $\theta = 0$ follows from the equalities

$$J_-(x,0) = J_+(x,0) = \frac{1}{\sqrt{2\pi x}}\,.$$

4.1.4 A general remark

Let us consider again formulas ($13_\pm$), and define

$$J(c,m) = J_+(c,m)+J_-(c,m) = \int \mathbf{n}(d\epsilon)\left(1-1\!\!1_{V(\epsilon)\le c}\exp[-\frac{m^2}{2}\int_0^{V(\epsilon)} du\, \exp(2\epsilon_u)]\right).$$

With the help of the formula

$$\left(\frac{mK_{\mu+1}(m)}{K_\mu(m)} + \frac{mI_{\mu-1}(m)}{I_\mu(m)} - 2\mu\right)^{-1} = I_\mu(m)\, K_\mu(m)\,,$$

(see, e.g., bottom of p. 29 in Yor (1993)) we obtain

$$\int_0^\infty dc \exp(-\frac{\mu^2 c}{2})\, J(c,m) = \frac{1}{\mu^2 I_\mu(m) K_\mu(m)}\,. \tag{15}$$

On the other hand (cf. Yor (1993), formula (17)), we have

$$\frac{\mu^2}{2}\int_0^\infty dt\, \exp(-\frac{\mu^2 t}{2})\, E[\exp(-\frac{m^2}{2}A_{g_t})] = 2\mu I_\mu(m) K_\mu(m)\,. \tag{16}$$

Therefore,

$$\left(\frac{\mu^2}{2}\int_0^\infty dt\, \exp(-\frac{\mu^2 t}{2})\, E[\exp(-\frac{m^2}{2}A_{g_t})]\right)\left(\frac{\mu^2}{2}\int_0^\infty dc\, \exp(-\frac{\mu^2 c}{2})\, J(c,m)\right) = \mu\,.$$

The purpose of the next lines is to show (quite simply, indeed) that this identity holds in a general setting.

Lemma 1 *For any Borel function $f : I\!\!R \to I\!\!R_+$,*

$$\frac{\mu^2}{2}\int_0^\infty dt\, \exp(-\frac{\mu^2 t}{2})\, E[\exp(-A^f_{g_t})] = \mu\int_0^\infty ds\, E[\exp -(A^f_{\tau_s} + \frac{\mu^2}{2}\tau_s)]\,.$$

This equality follows directly from excursion theory. Nevertheless, instead of giving an excursion theoretical proof, we present a proof based on "balayage" formula with the help of the following lemma.

Lemma 2 *Let $(z_t, t\ge 0)$ be a positive $(\mathcal{F}_t)$-predictable process and S_k an exponential variable with parameter $k^2/2$, independent of $(\mathcal{F}_t)$. Then,*

$$E(z_{g_{S_k}}) = kE\left(\int_0^\infty ds \exp(-\frac{k^2}{2}\tau_s) z_{\tau_s}\right).$$

Proof of lemma 2: We assume that z is bounded. Then, from the balayage formula

$$\begin{aligned} E\left(z_{g_{S_k}}|B_{S_k}|\right) &= E\left(\int_0^{S_k} d\ell_u\, z_u\right) = E\left(\int_0^{\ell_{S_k}} ds\, z_{\tau_s}\right) \\ &= E\left(\int_0^\infty ds\, 1\!\!1_{S_k\ge\tau_s}\, z_{\tau_s}\right), \end{aligned}$$

and it follows that

$$E\Big(z_{g_{S_k}}|B_{S_k}|\Big) = E\Big(\int_0^\infty ds\, z_{\tau_s}\, \exp\Big(-\frac{k^2}{2}\tau_s\Big)\Big)\,.$$

Since $|B_{S_k}|$ is independent of $\mathcal{F}_{g_{S_k}}$ and $|B_{S_k}| \stackrel{law}{=} \ell_{S_k}$ with common exponential distribution with parameter k, the left-hand side of the previous equality equals

$$E\Big(z_{g_{S_k}}\Big)E\Big(|B_{S_k}|\Big) = E\Big(z_{g_{S_k}}\Big)E(\ell_{S_k}) = \frac{1}{k}E\Big(z_{g_{S_k}}\Big)\,.$$

The result follows. □

Proof of lemma 1: It follows from lemma 2 by taking $z_t = \exp -A_t^f$. □

Recall that from (2)

$$J(c,f) = \int \mathbf{n}(d\epsilon)\left(1 - 1\!\!1_{V(\epsilon)\leq c}\, \exp -A_V^f(\epsilon)\right).$$

The Laplace transform of J is given by

$$\frac{\mu^2}{2}\int_0^\infty dc \exp(-\frac{\mu^2 c}{2})\, J(c,f) = \int \mathbf{n}(d\epsilon)\left(1 - \exp -(A_V^f(\epsilon) + \frac{\mu^2}{2}V(\epsilon))\right). \quad (17)$$

On the other hand, from the master multiplicative formula

$$\int_0^\infty ds\, E[\exp -(A_{\tau_s}^f + \frac{\mu^2}{2}\tau_s)] = \left[\int \mathbf{n}(d\epsilon)\left(1 - \exp -(A_V^f(\epsilon) + \frac{\mu^2}{2}V(\epsilon))\right)\right]^{-1}, \quad (18)$$

so that the comparison between (18) and (17) yields

$$\left(\frac{\mu^2}{2}\int_0^\infty dt\, \exp(-\frac{\mu^2 t}{2})\, E[\exp(-A_{g_t}^f)]\right)\left(\frac{\mu^2}{2}\int_0^\infty dc\, \exp(-\frac{\mu^2 c}{2})\, J(c,f)\right) = \mu\,.$$

4.1.5 Separating positive and negative excursions

We now go back to the initial case of Parisian options, where we consider only excursions above level 0.
For the reader's convenience, we recall that

$$H_{c,d} \stackrel{def}{=} \inf\{s : 1\!\!1_{B_s^{(\nu)}>0}(s-g_s) \geq c \quad \text{or} \quad 1\!\!1_{B_s^{(\nu)}<0}(s-g_s)\geq d\}\,,$$

and

$$H_c^+ = H_{c,\infty} = \inf\{s : 1\!\!1_{B_s^{(\nu)}>0}(s-g_s)\geq c\}\,.$$

Proposition 1 (i) extends to the stopping time H_c^+ :

Proposition 8 *The process $\{B_{g_{H_c^+}} + u\,, u \leq c\}$ is a Brownian meander with length c, independent of the σ-field $\mathcal{F}^+_{g_{H_c}}$.*

Our aim is to compute

$$u^+(\alpha, k, \theta; c) \stackrel{def}{=} E\Big[\exp(\alpha B_{H_c^+} - \frac{k^2}{2} H_c^+ - \frac{\theta^2}{2} A_{H_c^+})\Big]$$

which, from Proposition 8, is equal to :

$$u^+(\alpha, k, \theta; c) = u_b^+(k, \theta; c)\ \exp(-\frac{k^2 c}{2})\ u_m^+(\alpha, \theta; c)\,,$$

where

$$u_b^+(k, \theta; c) \stackrel{def}{=} E[\exp - \frac{1}{2}(k^2 g_{H_c^+} + \theta^2 A_{g_{H_c^+}})]$$

and

$$u_m^+(\alpha, \theta; c) \stackrel{def}{=} E\Big[\exp(\alpha B_{H_c^+} - \frac{\theta^2}{2} \int_{g_{H_c^+}}^{H_c^+} du\ \exp(2B_u))\Big]\,.$$

The corresponding $(JL)_m^{1+}$ problem reduces to the computation of

$$E[\exp(\alpha \sqrt{c}\, m_1 - \frac{\theta^2 c}{2} \int_0^1 du\ \exp(2\sqrt{c}\, m_u))]\,,$$

which is done in subsection 4.1.1. Thus, we are looking at the $(JL)_b^{1+}$ problem.

Taking care to the positive (resp. negative) excursions, the proof of Proposition 1 leads us to the $(\mathcal{F}_{g_t}^+)$-martingales

$$\mu_t^{\pm}\, z_{g_t} - \frac{1}{2} \int_0^t d\ell_u z_u\,,$$

where (z_t) is any bounded $(\mathcal{F}_t)$-previsible process, and (μ_t^+) (resp. (μ_t^-)) is the positive (resp. negative) part of Azéma's martingale. We now use the $(\mathcal{F}_{g_t}^+)$-stopping time $H_{c,d}$, and, since $\mu_{H_{c,d}}^+ = 0$ if $H_{c,d} = H_d^-$, we obtain

$$\begin{aligned} \sqrt{\frac{\pi c}{2}} E\Big(z g_{H_{c,d}} \mathbb{1}_{H_c^+ = H_{c,d}}\Big) &= \frac{1}{2} E\Big(\int_0^{H_{c,d}} d\ell_u z_u\Big) \\ \sqrt{\frac{\pi d}{2}} E\Big(z g_{H_{c,d}} \mathbb{1}_{H_d^- = H_{c,d}}\Big) &= \frac{1}{2} E\Big(\int_0^{H_{c,d}} d\ell_u z_u\Big)\,. \end{aligned}$$

The usual change of time yields

$$E\Big(\int_0^{H_{c,d}} d\ell_u z_u\Big) = \int_0^\infty ds E\Big(z_{\tau_s} \mathbb{1}_{\Delta^+(\tau_s) \le c}\ \mathbb{1}_{\Delta^-(\tau_s) \le d}\Big)\,,$$

where $\Delta^{\pm}(\tau_s)$ is the maximum length of positive (resp. negative) excursions up to time τ_s.
We now apply the above computation to $z_u = \exp(-A_u^f)$.
The multiplicative master formula implies that

$$E\Big(z_{\tau_s} \mathbb{1}_{\Delta^+(\tau_s) \le c}\ \mathbb{1}_{\Delta^-(\tau_s) \le d}\Big) = \exp(-sK(c, d; f))\,,$$

where

$$K(c,d;f) \stackrel{def}{=} \int \mathbf{n}(d\epsilon)\Big(1 - 1\!\!1_{V^+(\epsilon)\le c,\, V^-(\epsilon)\le d} \exp(-A_V^f(\epsilon))\Big)$$

and V^+ (resp. V^-) is the lifetime for positive (resp. negative) excursions. We split $\mathbf{n}$ into $\mathbf{n}_+$ and $\mathbf{n}_-$; thus,

$$K(c,d;f) = \int \mathbf{n}_+(d\epsilon)\Big(1 - 1\!\!1_{V(\epsilon)\le c} \exp(-A_V^f(\epsilon))\Big) + \int \mathbf{n}_-(d\epsilon)\Big(1 - 1\!\!1_{V(\epsilon)\le d} \exp(-A_V^f(\epsilon))\Big).$$

It follows that

$$\begin{aligned} E(1\!\!1_{H_c^+ = H_{c,d}} \exp(-A^f_{g_{H_{c,d}}})) &= \frac{1}{\sqrt{2\pi c}\, K(c,d;f)} \\ E(1\!\!1_{H_d^- = H_{c,d}} \exp(-A^f_{g_{H_{c,d}}})) &= \frac{1}{\sqrt{2\pi d}\, K(c,d;f)}, \end{aligned}$$

so, adding the members of these equalities, we get

$$E(\exp(-A^f_{g_{H_{c,d}}})) = \frac{1}{K(c,d;f)}\Big(\frac{1}{\sqrt{2\pi c}} + \frac{1}{\sqrt{2\pi d}}\Big).$$

It remains to observe that

$$K(c,d;f) = J_+(c,f) + J_-(d,f).$$

In particular, the value of u_b^+ is obtained by choosing $f(x) = \dfrac{1}{2}(k^2 + \theta^2 \exp(2x))$ and letting $d \to \infty$.

4.2 On problem $(JL)^2$

We return to the general case of a Brownian motion with drift equal to ν ; we recall that here $\Sigma = T_{-a,b}$. Let us denote $m = \exp(-a)$ and $M = \exp b$, and assume that $m < M$. We introduce the stopping times for the Bessel process $T_\rho = \inf\{t \ge 0 \mid R_t^{(\nu)} = \rho\}$ and we denote $T^* = T_m \wedge T_M$.

From the representation (6), and from (7), we deduce that, in this case,

$$(\Sigma, \exp B_\Sigma^{(\nu)}, A_\Sigma^{(\nu)}) \stackrel{(law)}{=} (C_{T^*}^{(\nu)}, R_{T^*}^{(\nu)}, T^*).$$

The strong Markov property and the following proposition (see Revuz-Yor, chap. XI, 1.22 p 433 for a proof) will lead us to the solution.

Proposition 9 *Let $\alpha > 0$ and T be an $(\mathcal{R}_s)$-stopping time such that $E_\alpha^{(0)}(T^\nu) < \infty$ for any $\nu > 0$. Then, for each $\mu, \nu \ge 0$, and for each $(\mathcal{R}_{T+})$-measurable random variable $Y \ge 0$,*

$$E_\alpha^{(\nu)}[Y(\frac{\alpha}{R_T})^\nu \exp -\frac{k^2}{2} C_T] = E_\alpha^{(\lambda)}[Y(\frac{\alpha}{R_T})^\lambda], \tag{19}$$

where $\lambda = \sqrt{\nu^2 + k^2}$.

The strong Markov property enables us to compute

$$E_\alpha^{(\nu)}[\exp(-\frac{\theta^2}{2}T^*)\,,R_{T^*}=m]$$

using the Laplace transforms of T_m and T_M which are obtained from (8), namely,

$$E_\alpha^{(\nu)}[\exp(-\frac{\theta^2}{2}T^*)\,,R_{T^*}=m]=(\frac{m}{\alpha})^\nu\frac{I_\nu(M\theta)K_\nu(\alpha\theta)-I_\nu(\alpha\theta)K_\nu(M\theta)}{I_\nu(M\theta)K_\nu(m\theta)-I_\nu(m\theta)K_\nu(M\theta)}\,.$$

In the same way, we obtain $E_\alpha^{(\nu)}[\exp(-\frac{\theta^2}{2}T^*)\,,R_{T^*}=M]$ by interverting the parameters m and M in the right-hand side. Then, from (19) (see also Pitman-Yor (1980))

$$\left(\frac{\alpha}{m}\right)^\nu E_\alpha^{(\nu)}[\exp(-\frac{k^2}{2}C_{T^*}-\frac{\theta^2}{2}T^*)\,,R_{T^*}=m]=\left(\frac{\alpha}{m}\right)^\lambda E_\alpha^{(\lambda)}[\exp(-\frac{\theta^2}{2}T^*)\,,R_{T^*}=m]\,,$$

which characterizes the Laplace transform of the joint law of $(C_{T^*}^{(\nu)},R_{T^*}^{(\nu)},T^*)$ and, since

$$E_0\Big(\exp(-\frac{k^2}{2}\Sigma-\frac{\theta^2}{2}A_\Sigma^{(\nu)})\,;B_\Sigma^{(\nu)}=-a\Big)=E_1^{(\nu)}[\exp(-\frac{k^2}{2}C_{T^*}-\frac{\theta^2}{2}T^*)\,,R_{T^*}=e^{-a}]\,,$$

we obtain

Proposition 10 *In the case $\Sigma=T_{-a,b}$, when $\exp(-a)<\exp b$, the joint law of $(\Sigma,\exp B_\Sigma^{(\nu)},A_\Sigma^{(\nu)})$ is given by*

$$\begin{aligned}&E_0\Big(\exp(-\frac{k^2}{2}\Sigma-\frac{\theta^2}{2}A_\Sigma^{(\nu)})\,;B_\Sigma^{(\nu)}=-a\Big)\\&\qquad=\exp(-\nu a)\frac{I_\lambda(e^b\theta)K_\lambda(\theta)-I_\lambda(\theta)K_\lambda(e^b\theta)}{I_\lambda(e^b\theta)K_\lambda(e^{-a}\theta)-I_\lambda(e^{-a}\theta)K_\lambda(e^b\theta)}\,,\end{aligned}$$

where $\lambda=\sqrt{\nu^2+k^2}$, and analogous formula for $B_\Sigma^{(\nu)}=b$.

These formulas are still valid in the case $\theta=0$, using equivalent expressions for the Bessel functions (See the Appendix).

5 A solution of problem (V)

5.1 The joint law of $(B_t^{(\nu)},A_t^{(\nu)})$

Thanks to formula (6), we obtain

$$v^{(\nu)}(\lambda;a,b,k)=E[\int_0^\infty\frac{du}{(R_u^{(\nu)})^2}\,\exp(-\lambda C_u^{(\nu)})\,(aR_u^{(\nu)}+bu-k)^+]\,.$$

Next, we use the absolute continuity relationship between the laws of different Bessel processes, i.e.,

$$P_\alpha^{(\nu)}|\mathcal{R}_t = \left(\frac{R_t}{\alpha}\right)^\nu \exp(-\frac{\nu^2}{2}\int_0^t \frac{ds}{R_s^2}) \cdot P_\alpha^{(0)}|\mathcal{R}_t\,,$$

which was already recalled in Proposition 9, and where $P_\alpha^{(\nu)}$ denotes the law (on the canonical space) of the Bessel process of index ν, starting at α. Thus if we define $\theta = \sqrt{2\lambda + \nu^2}$, we obtain

$$v^{(\nu)}(\lambda; a, b, k) = \int_0^\infty du\, E_1^{(\theta)}\Big[\frac{(aR_u + bu - k)^+}{R_u^{2+\theta-\nu}}\Big]\,,$$

In order to compute this quantity, we need to know

$$E_1^{(\theta)}(\frac{1\!\!1_{R_u > q}}{R_u^{2m}}) = \int_q^\infty \frac{d\rho}{u\rho^{2m}}\,\rho^{\theta+1}\exp(-\frac{1+\rho^2}{2u})\; I_\theta(\frac{\rho}{u})\,, \tag{20}$$

where $q = \dfrac{(k-bu)^+}{a}$ for $a > 0$. We have no closed form formulas for these integrals except in the Asian case, i.e., for $a = 0$, which simplifies the computation since $q = 0$. In that case, we have the alternative expression to (20)

$$\begin{aligned} E_1^{(\theta)}(\frac{1}{R_u^{2m}}) &= \frac{1}{\Gamma(m)}\int_0^{1/2u} dv\, e^{-v} v^{m-1}\,(1-2uv)^{\theta-m} \qquad (21)\\ &= \frac{1}{\Gamma(m)}\int_0^1 \frac{dw}{(2u)^m}\,\exp(-\frac{w}{2u})\, w^{m-1}(1-w)^{\theta-m} \\ &= \frac{\Gamma(1+\theta-m)}{\Gamma(1+\theta)\,(2u)^m}\Phi(m, 1+\theta; -\frac{1}{2u}) \end{aligned}$$

where $\Phi(\alpha,\gamma;z)$ denotes the hypergeometric function with parameters α and γ (see Lebedev (1972), p. 266); formula (21) is found in Yor (1993), p. 28, and the identity between (20) and (21) for $q = 0$, follows from exercise 12, p. 278 in Lebedev (1972) together with the relation $\Phi(\alpha,\gamma;z) = e^z\Phi(\gamma-\alpha,\gamma;-z)$.

In the case $a > 0, b > 0$, using obvious changes of variables, we obtain

$$v^{(\nu)}(\lambda; a, b, k) = a\int_0^\infty du\,\Theta(\nu, u) + b\int_0^\infty du\, u\,\Theta(\nu-1, u) - k\int_0^\infty du\,\Theta(\nu-1, u)\,,$$

where $$\begin{cases} \Theta(\nu, u) = \exp(-\frac{1}{2u})\displaystyle\int_0^\infty d\rho\,\rho^\nu \exp(-\frac{\rho^2}{2u}) I_\theta(\frac{\rho}{u}) & \text{for } u > \frac{k}{b} \\ \qquad\quad\;\; = \exp(-\frac{1}{2u})\displaystyle\int_{\frac{k-bu}{au}}^\infty d\rho\,\rho^\nu \exp(-\frac{\rho^2}{2u}) I_\theta(\frac{\rho}{u}) & \text{for } u < \frac{k}{b}\,. \end{cases}$$

In the case $u > \dfrac{k}{b}$, $\Theta(\nu, u)$ can be evaluated from (21), in the other case, we are

lead to use the incomplete Gamma function $\Gamma(x,\lambda) = \int_x^\infty dt\, e^{-t} t^{\lambda-1}$ and the series expansion of the Bessel function I_θ.
In the general case the integral may be evaluated, for some numerical values of the parameters, by computer system (e.g., Mapple or Mathematica).

5.2 Further research

To illustrate the flexibility of this method, we remark that it also allows to obtain the price of spread Asian options, or "Asian chooser" options, whose pay-off is $(aA_t^{(\nu)} + \tilde{a}\tilde{A}_t^{(\tilde{\nu})})$ where the two Asian payments are related to different underlying assets, with independent Brownian motions. In fact, our method enables to obtain the joint Laplace transform

$$v^{(\nu,\tilde{\nu})}(\lambda,\tilde{\lambda},a,\tilde{a},k) = E\Big(\int_0^\infty dt \int_0^\infty ds\, e^{-\lambda t} e^{-\tilde{\lambda} s}\Big(aA_t^{(\nu)} + \tilde{a}\tilde{A}_s^{(\tilde{\nu})} - k\Big)^+$$

$$= \int_0^\infty du \int_0^\infty dv\, E_1^{(\theta)}\Big(\frac{1}{R_u^{2+\theta-\nu}}\Big)\, E_1^{(\tilde{\theta})}\Big(\frac{1}{R_v^{2+\tilde{\theta}-\tilde{\nu}}}\Big)\,(au + \tilde{a}v - k)^+ \,,$$

hence, this double integral can be evaluated (at least, in principle ...) thanks to formula (21).

6 Conclusion

The probabilistic tools used in this paper (Bessel processes and Excursions theory) have been intensively studied by mathematicians for at least thirty years. Although fairly new in finance, their power appears in various applications such as exotic options pricing and interest rate derivative securities in particular in the Cox-Ingersoll-Ross framework.

The final numerical results associated with the formulas derived throughout the paper depend on the inversion of Laplace transforms. This problem has been solved in Geman and Eydeland (1995) for Asian options, in Geman-Yor (1995) for double barrier options. An approximation has been proposed for Parisian options in Cornwall et al. (1995).

As illustrated briefly at the end of section 5, the versatility of diffusion and excursion theories makes it possible to consider a much larger class of options; to illustrate further, our method could be extended to more general stopping times than $H_{c,d}$, e.g.,

$$H_\phi = \inf\{t : \mu_t^+ \geq \phi_+(\ell_t) \quad \text{or} \quad \mu_t^- \geq \phi_-(\ell_t)\}\,,$$

where μ_t denotes the Azéma martingale (related with $t - g_t$), (ℓ_t) is the local time of the Brownian motion at 0, and $\phi_\pm$ are Borel functions, but this is perhaps getting a little too far ahead from currently traded options.

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Appendix: some definitions

We recall here some definitions and results about Bessel Processes, Brownian meander and Excursion theory. For a precise study of Bessel processes and Brownian Excursions, the reader can refer to chapter XI and chapter XII of Revuz-Yor (1994); some results about the Brownian meander are given in exercise 3.8 chapter XII of that book. The seminal paper of Chung (1976) contains a number results about the Brownian meander. Most of the results about slow filtrations and "balayage" formulas are found in Dellacherie, Maisonneuve and Meyer (1992).

A.1 Bessel processes

A Bessel Process with index $\nu \geq 0$ ([7])is a diffusion process which takes values in $\mathbb{R}_+$ and has infinitesimal generator

$$\frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\frac{d}{dx} .$$

The number $d = 2(\nu+1)$ is called the dimension of the Bessel process.

The Bessel process with dimension d starting at α satisfies the equation

$$R_t = \alpha + B_t + \frac{d-1}{2}\int_0^t \frac{1}{R_s}\, ds \, ,$$

where B_t is a Brownian motion. On the canonical space $\Omega = C(\mathbb{R}_+, \mathbb{R}_+)$, we denote by R the canonical map $R_t(\omega) = \omega(t)$, by $\mathcal{R}_t = \sigma(R_s, s \leq t)$ the canonical filtration and by $P_\alpha^{(\nu)}$ (or ${}^{(d)}P_\alpha$) the law of the Bessel process of index ν (of dimension d), starting at α, i.e., such that $P_\alpha^{(\nu)}(R_0 = \alpha) = 1$. The Bessel process of index ν has a transition density defined by

$$p_t^{(\nu)}(\alpha, \rho) = \frac{\rho}{t}\left(\frac{\rho}{\alpha}\right)^\nu \exp\left(-\frac{\alpha^2+\rho^2}{2t}\right) I_\nu\left(\frac{\alpha\rho}{t}\right),$$

where I_ν is the usual modified Bessel function with index ν.

Both functions I_ν and K_ν satisfy the Bessel differential equation

$$x^2 u''(x) + x u'(x) - (x^2 + \nu^2) u(x) = 0$$

and are given by :

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{z^{2n}}{2^{2n}\, n!\, \Gamma(\nu+n+1)} \quad , \quad K_\nu(z) = \frac{\pi(I_{-\nu}(z) - I_\nu(z))}{2 \sin \pi\nu} .$$

[7]In this paper, when we consider a negative index ν, the corresponding Bessel process is only taken care of up to its first hitting time of zero. For simplicity, we do not discuss this case here.

In particular, $I_\nu(z) \overset{z\to 0}{\sim} \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu$ and $K_\nu(z) \overset{z\to 0}{\sim} \frac{\Gamma(\nu)}{2}\left(\frac{z}{2}\right)^{-\nu}$.

(More results on I_ν and K_ν are found in Lebedev (1972).) For each pair $(\nu,\theta) \in \mathbb{R}_+ \times \mathbb{R}$, and for each $t > 0$, one has

$$E_\alpha^{(\nu)}[\exp -\frac{\theta^2}{2}\int_0^t \frac{ds}{R_s^2}\,|R_t = \rho] = \frac{I_\lambda(\frac{\rho\alpha}{t})}{I_\nu(\frac{\rho\alpha}{t})},$$

where $\lambda = \sqrt{\nu^2+\theta^2}$.

A.2 Some σ-fields associated with random times

Let us denote by $(\mathcal{F}_t, t \geq 0)$ the natural filtration of the Brownian motion $(B_t, t \geq 0)$.
If L is an almost surely strictly positive random variable, we define the σ-field $\mathcal{F}_L^-$ of the past up to L as the σ-algebra generated by the variables ζ_L, where ζ is a predictable process.
Let t be given. The excursion which straddles t evolves between its left extremity $g_t = \sup\{s \leq t : B_s = 0\}$ and its right extremity $d_t = \inf\{s \geq t : B_s = 0\}$. Recall that g_t is not a $\mathcal{F}_t$-stopping time. We denote by $(\mathcal{F}_{g_t}^+, t \geq 0)$ the extended "slow Brownian filtration" $(\mathcal{F}_{g_t}^+ = \mathcal{F}_{g_t}^- \vee (\mathrm{sgn}(B_t), t \geq 0)$, a subfiltration of $\mathcal{F}_t$.

A.3 The Brownian meander

We denote by $g = g_1 = \sup\{s \leq 1 : B_s = 0\}$ the left extremity of the excursion which straddles time 1. The Brownian meander is defined as the process

$$m_u = \frac{1}{\sqrt{1-g}}|B_{g+u(1-g)}|\,;\; u \leq 1\,.$$

The process m is a Brownian scaled part of the (normalized) Brownian excursion which straddles time 1. The process m is independent of $\mathcal{F}_g^-$.

Using Brownian scaling, we remark that for fixed time t, the process

$$m_u^{(t)} = \frac{1}{\sqrt{t-g_t}}|B_{g_t+u(t-g_t)}|\,;\; u \leq 1$$

is a Brownian meander independent of the σ-field $\mathcal{F}_{g_t}^+$; in particular, the law of $m^{(t)}$ does not depend on t. For each $(\mathcal{F}_{g_t}^+)$-stopping time τ, in particular for $\tau = H_c$, $m^{(\tau)}$ is a meander. Remark that the definition of H_c implies that $H_c - g_{H_c} = c$.
The Azéma martingale $\mu_t \overset{def}{=} \mathrm{sgn}(B_t)\sqrt{\frac{\pi}{2}(t-g_t)}$ is the projection of B_t on the filtration $\mathcal{F}_{g_t}^+$. Let ℓ_t be the local time at 0 of the Brownian motion. The projection of $|B_t| - \ell_t$ on the filtration $(\mathcal{F}_{g_t}^+)$, and, in fact, also on the filtration $(\mathcal{F}_{g_t}^-)$ is equal to $|\mu_t| - \ell_t$.

A.4 The Itô measure of Brownian excursions

Let $(B_t, t \geq 0)$ be a Brownian motion and (τ_s) be the inverse of the local time (ℓ_t) at level 0. The set $\{\cup_{s\geq 0}]\tau_{s-}(\omega), \tau_s(\omega)[\,\}$ is (almost surely) equal to the complement of the zero set $\{u : B_u(\omega) = 0\}$. The excursion process $(e_s, s \geq 0)$ is defined as

$$e_s(\omega)(t) = \mathbb{1}_{\{t \leq \tau_s - \tau_{s-}\}} B_{\tau_{s-} + t}, t \geq 0.$$

This is a path-valued process $e : I\!\!R_+ \to \Omega_*$, where

$$\Omega_* = \{\epsilon : I\!\!R_+ \to I\!\!R : \exists V(\epsilon) < \infty, \text{ with } \epsilon(V(\epsilon) + t) = 0, \forall t \geq 0$$

$$\epsilon(u) \neq 0, \forall\, 0 < u < V(\epsilon),\ \epsilon(0) = 0,\ \epsilon \text{ is continuous}\}.$$

Hence, $V(\epsilon)$ is the lifetime of ϵ.

The excursion process is a Poisson Point Process, and $\mathbf{n}(\Gamma)$ is defined as the intensity of the Poisson process

$$N_t^\Gamma \stackrel{def}{=} \sum_{s \leq t} \mathbb{1}_{e_s \in \Gamma},$$

i.e., the positive real γ such that $N_t^\Gamma - t\gamma$ is an $(\mathcal{F}_{\tau_t})$-martingale.
The Itô-Williams' description of the measure $\mathbf{n}$ is

$$\mathbf{n}(d\epsilon) = \int_0^\infty \mathbf{n}_V(dv)\, \frac{1}{2}(\Pi_+^v + \Pi_-^v)\,(d\epsilon)$$

where $\mathbf{n}_V(dv) = \dfrac{dv}{\sqrt{2\pi v^3}}$ is the law of the lifetime V under $\mathbf{n}$ and Π_+^v (resp. Π_-^v) is the law of the Bessel Bridge (resp. the law of its opposite) with dimension 3 and length v.