Criteria of positivity for the density of the law of a Wiener functional

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ABSTRACT. We consider a function f on an abstract Wiener space, with values in $\mathbb{R}^d$, which is regular and non-degenerate in the sense of the Malliavin calculus. Its law admits a density $\mathbf{p}$ which is C^∞ on $\mathbb{R}^d$. We give criteria ensuring that $\mathbf{p}(\xi) > 0$ for a fixed ξ in $\mathbb{R}^d$. The first one is stated in terms of capacities $c_{r,p}$ on the Wiener space. The second one is related to some "restrictions" of f to subspaces of dimension d of the Cameron-Martin space. Finally, we apply this criterion to obtain another one, previously given by Aida-Kusuoka-Stroock, in the case where function f admits a "skeleton". The criteria are actually proved in the more general situation where the basic Wiener measure is replaced by a measure with a regular density.

1 Introduction

In several works, criteria of positivity of densities were studied for solutions of SDE's ([3], [1]) or SPDE's ([13]). These criteria are based on the existence of a "skeleton", which is a well defined "restriction" to the Cameron-Martin space which determines the function. A general version of this kind of criteria is given in [1] and, under a slightly different form, in [13].

In sections 2 and 3 of this paper, we are interested in criteria of positivity without any assumption of existence of a skeleton. Then, in section 4, we show that the criterion of section 3 allows us to obtain essentially the result of [1] just above mentioned.

The setting of this paper is that of an abstract Wiener space (E, H, μ), and we widely use, without defining them, the notions and notation of Malliavin's calculus (we refer, for example, to [15] and [4]). The characterizations that we have in mind are related to capacities on Wiener space, for which we refer to [11], [14], [7], [12],...

We consider, throughout all this work, an element $f = (f_1, \dots, f_d)$ in $(\mathbb{D}^\infty)^d$, where $\mathbb{D}^\infty$ denotes the set of real smooth functionals on E in the sense of Malliavin. We set $Jf = [\det(< Df_i, Df_j >_H)]^{1/2}$ and we assume the classical condition of non-degeneracy

$$\frac{1}{Jf} \in \cap_{p>1} L^p(\mu).$$

We also consider a non-negative function g in $\mathbb{D}^\infty$ and we denote by μ_g the measure with density g with repect to μ. It then is known that the law $\mu_g \circ f^{-1}$ of f under μ_g has a smooth density with respect to the Lebesgue measure on $\mathbb{R}^d$. In what follows, we denote this density by $\mathbf{p}_g$ (abreviated in $\mathbf{p}$ if $g = 1$).

If $r \geq 0$ and $p > 1$, there exists a capacity $c_{r,p}$ associated with the Sobolev space $\mathbb{D}_p^r$ on E (r is the "order of derivation" and p is the "power of integration"). Then f (resp. g) admits a μ-representative which is quasi-continuous with respect to all capacities $c_{r,p}$ (we shall say that it is ∞-quasi-continuous). Henceforth, f (resp. g) denotes "this" representative.

2 Positivity and capacities

Our goal, in this section, is to prove the following result:

Theorem 1 *Let $\xi \in \mathbb{R}^d$. The following properties are equivalent:*

(i) $\mathbf{p}_g(\xi) > 0$

(ii) $\exists r \geq 0 \ \ \exists p > 1 \quad c_{r,p}(\{f = \xi\} \cap \{g > 0\}) > 0$

(iii) $\forall r > 0$ *and* $\forall p > 1$ *such that* $rp > d$, $c_{r,p}(\{f = \xi\} \cap \{g > 0\}) > 0$.

We begin with a key lemma.

Lemma 1.1 *If $\mathbf{p}_g(\xi) = 0$, then, for all $n \in \mathbb{N}$, for all $r \in \mathbb{N}^d$ such that $|r| < n$, $\dfrac{\partial^{|r|}}{\partial x^r} \mathbf{p}_{g^n}(\xi) = 0$.*

The above result is proved, for $g = 1$, in [2]. The following proof follows the same ideas.

Proof: Let $r \in \mathbb{N}^d$. By [15, Lemma 2, p.54], there exists a linear functional

l_r from $\mathbb{D}^\infty$ into itself, such that, for any smooth function u on $\mathbb{R}^d$ with compact support and for any $h \in \mathbb{D}^\infty$,

$$\int \frac{\partial^{|r|}u}{\partial x^r} \circ f \; h \, d\mu = \int u \circ f \; l_r(h) \, d\mu.$$

By the explicit form of $l_r(h)$, one sees that $l_r(h)$ depends linearly on h and on its Malliavin derivatives up to order $|r|$. Hence, if $|r| < n$, there exists $h_{r,n} \in \mathbb{D}^\infty$ such that $l_r(g^n) = g \; h_{r,n}$. Therefore, if $|r| < n$, for any u as above,

$$\int u(x) \, \frac{\partial^{|r|}}{\partial x^r} \mathbf{p}_{g^n}(x) \, dx = (-1)^{|r|} \int u \circ f \; g \; h_{r,n} \, d\mu.$$

Now, by [15, Theorem 1.14, p.60] and the notation therein, the last integral may be written as $\int u(x) \; < \delta_x(f), g \; h_{r,n} > \; dx$. We therefore have

$$\frac{\partial^{|r|}}{\partial x^r} \mathbf{p}_{g^n}(\xi) = (-1)^{|r|} < \delta_\xi(f), g \; h_{r,n} > .$$

According to [14], $\delta_\xi(f)$ may also be viewed as a measure, the pairing corresponding to the integral of the ∞-quasi-continuous representative. Thus, if $\mathbf{p}_g(\xi) =< \delta_\xi(f), g >= 0$, then $g = 0$ $\delta_\xi(f)$-a.e. and therefore, $< \delta_\xi(f), g \; h_{r,n} >= 0$, which yields the desired result. 2

Lemma 1.2 *Assume* $\mathbf{p}_g(\xi) = 0$. *Then, for all* $r \in \mathbb{N}^d$, *for all* $i \geq 0$ *and* $a > 0$,

$$\lim_{\varepsilon \to 0} \varepsilon^{-i} \mathbb{E}[\prod_{j=1}^d |f_j - \xi_j|^{r_j} \exp(-a\frac{|f-\xi|^2}{\varepsilon^2}) g^n] = 0$$

for n *big enough.*

Proof: We have

$$\mathbb{E}[\prod_{j=1}^d |f_j - \xi_j|^{r_j} \exp(-a\frac{|f-\xi|^2}{\varepsilon^2}) g^n]$$

$$= \int \prod_{j=1}^d |x_j - \xi_j|^{r_j} \exp(-a\frac{|x-\xi|^2}{\varepsilon^2}) \mathbf{p}_{g^n}(x) \, dx.$$

Denote by φ_ε the indicator of $\{x; \; |x-\xi|^2 \leq \varepsilon\}$. We have

$$\int (1 - \varphi_\varepsilon(x)) \prod_{j=1}^d |x_j - \xi_j|^{r_j} \exp(-a\frac{|x-\xi|^2}{\varepsilon^2}) \mathbf{p}_{g^n}(x) \, dx$$

$$\leq \mathbb{E}[\prod_{j=1}^{d} |f_j - \xi_j|^{r_j} g^n] \exp(\frac{-a}{\varepsilon}),$$

and, by Taylor's formula and lemma 1.1,

$$\int \varphi_\varepsilon(x) \prod_{j=1}^{d} |x_j - \xi_j|^{r_j} \exp(-a\frac{|x-\xi|^2}{\varepsilon^2}) \mathbf{p}_{g^n}(x)\, dx$$

$$\leq (n!)^{-1} \sup_{|x-\xi|\leq\varepsilon} |\nabla^n \mathbf{p}_{g^n}(x)| \int \varphi_\varepsilon(x)|x-\xi|^{|r|+n} dx$$

$$\leq (n!)^{-1} C_d \sup_{|x-\xi|\leq 1} |\nabla^n \mathbf{p}_{g^n}(x)|\ \varepsilon^{(|r|+n+d)/2},$$

where C_d denotes the volume of the unit ball in $\mathbb{R}^d$. The result follows directly. 2

We then deduce the implication $(ii) \Longrightarrow (i)$ in theorem 1:

Corollary 1.2.1 *If* $\mathbf{p}_g(\xi) = 0$*, then* $c_{r,p}(\{f = \xi\} \cap \{g > 0\}) = 0$ *for all* $r \geq 0$ *and* $p > 1$*.*

Proof: Let $r \geq 0$ and $p > 1$. Suppose $\mathbf{p}_g(\xi) = 0$ and let $F_\varepsilon = \exp(-\frac{|f-\xi|^2}{\varepsilon^2})$. Then F_ε is ∞-quasi-continuous and belongs to $\mathbb{D}^\infty$. Let D be the derivative operator (from $\mathbb{D}^\infty$ into $\mathbb{D}^\infty(H)$). Let k be an integer greater than r. Then, by lemma 1.2 (and the Schwartz inequality), there exists $n \in \mathbb{N}$ such that, for $0 \leq k' \leq k$,

$$\lim_{\varepsilon\to 0} \|D^{k'}(F_\varepsilon g^n)\|_{L^p(\mu)} = 0,$$

and, therefore, $F_\varepsilon g^n$ tends to 0 in $\mathbb{D}_p^r$ as ε tends to 0. Since, for $a > 0$, $F_\varepsilon g^n \geq a^n$ on $\{f = \xi\} \cap \{g \geq a\}$, we get (cf. for example [10]) $c_{r,p}(\{f = \xi\} \cap \{g \geq a\}) = 0$. The result then follows, letting a tend to 0. 2

The implication $(iii) \Longrightarrow (ii)$ is obvious. Let us then show the implication $(i) \Longrightarrow (iii)$. Let us first recall that, by [16], the distribution $\delta_\xi(f)$ belongs to the dual space of $\mathbb{D}_p^r$ for any $r > 0$ and $p > 1$ such that $rp > d$. Assume that r and p are thus fixed. As a measure, $\delta_\xi(f)$ is then an (r,p)-finite energy measure and, therefore, it does not charge the (r,p)-polar sets (see [14], [12]). On the other hand, it is easy to prove, by approximation of this measure, that $< \delta_\xi(f), |f-\xi|^2 >= 0$ and therefore measure $\delta_\xi(f)$ does not charge the set $\{f \neq \xi\}$. Consequently, if $c_{r,p}(\{f = \xi\} \cap \{g > 0\}) = 0$, then measure $\delta_\xi(f)$ does not charge the set

$\{g > 0\}$. Therefore $\mathbf{p}_g(\xi) =< \delta_\xi(f), g >= 0$ and property (i) would not be satisfied. Theorem 1 is then proved.

Remark In [8], the authors built a "Gaussian Hausdorff measure on E of codimension d", denoted by ρ_d. As a consequence of their theorem 21, the equality $\mathbf{p}_g(\xi) = \int_{\{f=\xi\}} g\ (Jf)^{-1}\ d\rho_d$ holds (where Jf denotes the representative defined in lemma 3.1 below). It follows that $\mathbf{p}_g(\xi) > 0$ if and only if $\rho_d(\{f = \xi\} \cap \{g > 0\}) > 0$, which implies, by [8, theorem 9], property (iii) in the above theorem. This is another proof of the implication $(i) \Longrightarrow (iii)$, which needs actually less restrictive assumptions on f and g and was communicated to us by D. Feyel.

Clearly, as a consequence of the previous theorem, we have in particular that $\mathbf{p}(\xi) > 0$ if and only if $c_{d,2}(\{f = \xi\}) > 0$. To illustrate this criterion, we shall give another proof of a result due to S. Fang ([5]).

Theorem 2 *Suppose $d = 1$. Then $\{\mathbf{p} > 0\}$ is the interior of the support of $\mathbf{p}$.*

Proof: Let ξ be in the interior of the support of $\mathbf{p}$. Then $\mu(\{f < \xi\}) > 0$ and $\mu(\{f > \xi\}) > 0$. By ergodicity of the Ornstein-Uhlenbeck semi-group, one sees that, if (X_t) is the Ornstein-Uhlenbeck process in E with μ as initial law, then $\mathbb{P}(f(X_0) < \xi \text{ and } f(X_t) > \xi) > 0$ for t big enough. Now, as f is quasi-continuous, almost-surely the function $t \longrightarrow f(X_t)$ is continuous ([9]). Therefore $\mathbb{P}(\exists t\ \ f(X_t) = \xi) > 0$. This is equivalent ([9]) to $c_{1,2}(\{f = \xi\}) > 0$ and hence, by theorem 1, $\mathbf{p}(\xi) > 0$ holds. 2

Remarks

1) The above result is no longer true for $d \geq 2$ (see [1, Example 3.45]). D. Nualart gave recently another simpler example.

2) The result also is generally false, if $d = 1$, if $\mathbf{p}$ is replaced by $\mathbf{p}_g$.

3) D. Nualart gave recently a very short, analytic proof of the previous theorem under much weaker assumptions on f.

3 Positivity and restrictions to d-dimensional subspaces of H

Before stating the main result of this section, we introduce some notation. Since function f is ∞-quasi-continuous, we know according to [7] that, for any finite dimensional subspace K of H, there exists a slim set E_K in E (i.e., $\forall r, p\ \ c_{r,p}(E_K) = 0$) such that, for any $\omega \notin E_K$, the function

$$f_\omega^K : \ \ x \in K \longrightarrow f(\omega + x)$$

is C^∞ on K. We can define similarly the C^∞-function g_ω^K on K, for ω out of a slim set. If the dimension of K is d, we shall denote by $J(f_\omega^K)$ the absolute value of the determinant of the Jacobian matrix of f_ω^K. In what follows, ∞-quasi-everywhere means out of a slim set.
We fix a complete orthonormal basis, $\{h_0, h_1, \cdots\}$, of H. Denote by Θ the set of subsets of $\mathbb{N}$ containing d elements. For $\theta \in \Theta$, K_θ denotes the space spanned by $\{h_i;\ i \in \theta\}$. In what follows, we often replace in the notation K_θ simply by θ. For example, f_ω^θ is set for $f_\omega^{K_\theta}$. We then have the following result.

Theorem 3 *Let $\xi \in \mathbb{R}^d$. Then $\mathbf{p}_g(\xi) > 0$ if and only if there exists $\theta \in \Theta$ such that*

$$\mu(\{\omega;\ \exists x \in K_\theta\ \ f_\omega^\theta(x) = \xi,\ g_\omega^\theta(x) > 0 \text{ and } J(f_\omega^\theta)(x) > 0\}) > 0.$$

We begin with a few lemmas.

Lemma 3.1 *There exists an ∞-quasi-continuous representative of Jf, still denoted by Jf, and $Jf > 0$ ∞-quasi-everywhere.*

Proof: It is well-known that Jf and $(Jf)^{-1}$ belong to $\mathbb{D}^\infty$. Denoting in the same way the ∞-quasi-continuous representatives, we have $(Jf)(Jf)^{-1} = 1$ μ-a.s. and therefore, by a classical result, $(Jf)(Jf)^{-1} = 1$ ∞-quasi-everywhere. The result follows. 2

For $1 \le i \le n$ and $h \in H$, we denote by $D_h f_i$ the ∞-quasi-continuous representative of $< Df_i, h >_H$. If $\theta \in \Theta$, we set

$$J_\theta f = \left|\det(D_{h_k} f_i)_{1 \le i \le d, k \in \theta}\right|^{1/2}.$$

Lemma 3.2 *The set $\cap_{\theta\in\Theta}\{J_\theta f = 0\}$ is a slim set.*

Proof: Clearly, $\{Jf > 0\} = \cup_{\theta\in\Theta}\{J_\theta f > 0\}$ up to a slim set. The result follows then from lemma 3.1. 2

It should be noticed that, if K is a finite dimensional subspace of H, for ∞-quasi-every $\omega \in E$ and for all h and $x \in K$, $D_h f(\omega + x) = \partial_h f_\omega^K(x)$ where ∂_h denotes the usual derivative in direction h. Therefore, for ∞-quasi-every ω, for all $\theta \in \Theta$, $(J_\theta f)_\omega^\theta = J(f_\omega^\theta)$ on K_θ.

Lemma 3.3 *Let φ be a smooth function from $\mathbb{R}^d$ into $\mathbb{R}^d$ and let l be a non-negative continuous function on $\mathbb{R}^d$. Then, for $\xi, z \in \mathbb{R}^d$,*

$$\liminf_{r\to 0} \sum_{\varphi(x)=\xi+rz, J(\varphi)(x)>0} \frac{l(x)}{J(\varphi)(x)} \ge \sum_{\varphi(x)=\xi, J(\varphi)(x)>0} \frac{l(x)}{J(\varphi)(x)}.$$

Proof: It is an easy consequence of the inverse mapping theorem. 2

Set, for $x \in H$, $q(x) = (2\pi)^{-d/2} \exp(-\frac{|x|_H^2}{2})$. We denote, for $\theta \in \Theta$, by ω_θ the pseudo-orthogonal projection of $\omega \in E$ onto K_θ.

Lemma 3.4 *Let $\xi \in \mathbb{R}^d$ and $\theta \in \Theta$. Then,*

$$\mathbf{p}_g(\xi) \geq \int \sum_{f_\omega^\theta(x)=\xi, J(f_\omega^\theta)(x)>0} \frac{g_\omega^\theta(x)\ q(\omega_\theta + x)}{J(f_\omega^\theta)(x)}\ d\mu(\omega).$$

Proof: Let $B(\xi, r)$ be the open ball in $\mathbb{R}^d$ with center ξ and radius r and let C_d denote the volume of the unit ball in $\mathbb{R}^d$. Then, by the quasi-invariance of measure μ,

$$\mathbf{p}_g(\xi) = \lim_{r\to 0} \frac{1}{C_d r^d} \int 1_{B(\xi,r)} \circ f\ g\ d\mu$$

$$= \lim_{r\to 0} \frac{1}{C_d r^d} \int \int_{E\times K_\theta} 1_{B(\xi,r)} \circ f(\omega + x)\ g(\omega + x)\ q(\omega_\theta + x)\ d\mu(\omega)\ dx$$

$$\geq \liminf_{r\to 0} \frac{1}{C_d r^d} \int \int_{E\times K_\theta} 1_{B(\xi,r)} \circ f(\omega+x)\ g(\omega+x)\ q(\omega_\theta+x)\ 1_{\{J(f_\omega^\theta)(x)>0\}}\ d\mu(\omega)\ dx.$$

Using the change of variables formula (cf. [6]), we obtain

$$\mathbf{p}_g(\xi) \geq \liminf_{r\to 0} \frac{1}{C_d r^d} \int \int_{E\times B(\xi,r)} \sum_{f_\omega^\theta(x)=y, J(f_\omega^\theta)(x)>0} \frac{g_\omega^\theta(x)\ q(\omega_\theta + x)}{J(f_\omega^\theta)(x)}\ d\mu(\omega)\ dy.$$

The result follows then from lemma 3.3, using Fatou's lemma. 2

The previous lemma shows immediately that the condition in the statement of theorem 3 is sufficient. Suppose now $\mathbf{p}_g(\xi) > 0$. We have seen that $\delta_\xi(f)$ is a non-nul $(d,2)$-finite energy measure and $\delta_\xi(f)(\{g > 0\}) > 0$. According to lemma 3.2, there exist $\theta \in \Theta$ and $\varepsilon > 0$ such that

$$\delta_\xi(f)(\{J_\theta f \geq \varepsilon\} \cap \{g \geq \varepsilon\}) > 0.$$

Let φ be a real smooth function on $\mathbb{R}$ satisfying

$$0 \leq \varphi \leq 1,\ \varphi = 0 \text{ on }]-\infty, \varepsilon^2/4] \text{ and } \varphi = 1 \text{ on } [\varepsilon^2, +\infty[.$$

Then $\varphi((J_\theta f)^2) \in \mathbb{D}^\infty$ and $1_{\{J_\theta f \geq \varepsilon\}} \leq \varphi((J_\theta f)^2) \leq 1_{\{J_\theta f \geq \varepsilon/2\}}$, and similarly for $\varphi(g^2)$. Let (F_n) be an increasing sequence of closed sets such that $c_{d,2}(F_n^c) \downarrow 0$ and let, for any n, φ_n be the $(d,2)$-equilibrium potential of F_n^c (cf. [14] or [12]). Then,

$$\liminf_{r\to 0} \frac{1}{C_d r^d} \int_{F_n} 1_{B(\xi,r)} \circ f\ \varphi((J_\theta f)^2)\ \varphi(g^2)\ d\mu$$

$$\geq < \delta_\xi(f), \varphi((J_\theta f)^2)\, \varphi(g^2) > -C\|\varphi_n\|_{\mathbb{D}_2^d}$$
$$\geq \delta_\xi(f)(\{J_\theta f \geq \varepsilon\} \cap \{g \geq \varepsilon\}) - Cc_{d,2}(F_n^c)$$

where C does not depend on n. Hence, for n big enough,

$$\liminf_{r\to 0} \frac{1}{C_d r^d} \int_{F_n} 1_{B(\xi,r)} \circ f \; 1_L \; d\mu > 0,$$

where L denotes the set $\{J_\theta f \geq \varepsilon/2\} \cap \{g \geq \varepsilon/2\}$. Notice that, by [7], if ϕ is an ∞-quasi-continuous function and if K is a finite dimensional subspace of H, the function $\omega \in E \longrightarrow \phi_\omega^K$ is ∞-quasi-continuous as a function from E into $\mathcal{C}(K)$. It follows that

$$\gamma_\theta(\omega) = \sup\{|(D_{h_k} D_{h_l} f_j)_\omega^\theta(x)|;\ k,l \in \theta,\ 1 \leq j \leq d,\ x \in K_\theta \text{ and } |x|_H \leq 1\}$$

is ∞-quasi-continuous. We deduce from this result and from the tightness of capacity $c_{d,2}$ (cf. for example [7]) that there exists a compact set F in E, such that γ_θ and, for $k \in \theta$ and $1 \leq j \leq d$, $D_{h_k} f_j$ are continuous on F, and satisfying

$$\liminf_{r\to 0} \frac{1}{C_d r^d} \int_{F} 1_{B(\xi,r)} \circ f \; 1_L \; d\mu > 0.$$

By the remark following lemma 3.2,

$$\frac{1}{C_d r^d} \int_{F} 1_{B(\xi,r)} \circ f \; 1_L \; d\mu$$

$$= \int \left[\frac{1}{C_d r^d} \int 1_{B(\xi,r)} \circ f_\omega^\theta(x) 1_F(\omega + x) 1_{L_\omega^\theta}(x) q(\omega_\theta + x)\; dx\right] \; d\mu(\omega),$$

where L_ω^θ denote $\{x \in K_\theta;\ \ J(f_\omega^\theta)(x) \geq \varepsilon/2 \text{ and } g_\omega^\theta(x) \geq \varepsilon/2\}$. Denote by $I(r,\omega)$ the quantity between brackets. We suppose that the condition in the statement of the theorem is not satisfied. In other words, we suppose that, for μ-almost every ω, $\{x \in K_\theta;\ f_\omega^\theta(x) = \xi,\ g_\omega^\theta(x) > 0 \text{ and } J(f_\omega^\theta)(x) > 0\}$ is empty. Then, for μ-almost every ω, $f_\omega^\theta(\{x \in K_\theta;\ \omega + x \in F \text{ and } x \in L_\omega^\theta\})$ is a compact in $\mathbb{R}^d$ which does not contain ξ. Therefore, for μ-almost every ω, $I(r,\omega)$ is equal to 0 if r is small enough. Hence, to come to a contradiction, we have to dominate $I(r,\omega)$ independently of r and ω. Now, using again the change of variables formula,

$$I(r,\omega) = \frac{1}{C_d r^d} \int_{B(\xi,r)} \sum_{f_\omega^\theta(x) = y,\, x \in L_\omega^\theta,\, \omega + x \in F} \frac{q(\omega_\theta + x)}{J(f_\omega^\theta)(x)} \; dy.$$

Let $L(y,\omega)$ be the quantity under the integral. We have

$$L(y,\omega) \leq \frac{2}{\varepsilon} \sum_{f^\theta_\omega(x)=y, J(f^\theta_\omega)(x)\geq \varepsilon/2, \omega+x\in F} q(\omega_\theta + x).$$

As functions γ_θ and $D_{h_k} f_j$ are continuous on F compact and therefore bounded, by the inverse mapping theorem there exists some $\rho > 0$, independent of ω and y, such that, if x and x', $x \neq x'$, belong to the summation set, then $|x - x'|_H \geq \rho$. It is then easy to see that there exists a constant C depending only on d such that $L(y,\omega) \leq C\varepsilon^{-1}\rho^{-d}$. 2

4 Positivity and skeleton

In this section, we shall deduce from theorem 3 a characterization obtained in [1]. The hypothesis assumed by these authors is a hypothesis of existence of a skeleton in a meaning that we shall precise.

We denote, for $n \in \mathbb{N}$, by π_n the pseudo-orthogonal projection from E onto space H_n spanned by $\{h_i;\ 0 \leq i \leq n\}$. We may suppose that π_n is ∞-quasi-continuous and admits a "restriction" to H which is the true orthogonal projection from H onto H_n.

Let $\theta \in \Theta$. If φ is a real continuous function on K_θ, we set

$$m_\theta(\varphi)(x) = \sup_{t\in K_\theta, |t|_H \leq 1} |\varphi(x+t)|.$$

For a real C^2-function φ in K_θ, we set

$$l_\theta(\varphi)(x) = \sup\{|\varphi(x)|, |\partial_{h_k}\varphi(x)|, m_\theta(\partial_{h_k}\partial_{h_l}\varphi)(x);\ k,l \in \theta\}.$$

If φ is $\mathbb{R}^d$-valued, we set $l_\theta(\varphi) = \sup_{1\leq j\leq d} l_\theta(\varphi_j)$.

We set, for $\omega \in E$, $\omega_n = \pi_n(\omega)$ and $\omega'_n = \omega - \omega_n$. By [7], for μ-almost every ω, $f^{H_n}_{\omega'_n}$ is C^∞ on H_n. It follows that, if K is a subspace of H_n, then, for μ-almost every ω and all $h \in H_n$, $f^K_{h+\omega'_n}$ is C^∞ on K. The similar properties also hold for g. We shall denote $\pi_n h + \omega'_n$ by $(h,\omega)_n$.

Definition Function f will be said to be *s-regular* if there exists an $\mathbb{R}^d$-valued C^2-function $\tilde{f}$ on H such that

(S_1) $\forall \theta \in \Theta$, for μ-almost every ω,

$$\lim_{n\to\infty} l_\theta(f^\theta_\omega - \tilde{f}^\theta_{\omega_n})(x) = 0 \text{ for all } x \text{ in } K_\theta$$

(where, for all $h \in H$ and for all $x \in K_\theta$, $\tilde{f}^\theta_h(x) = \tilde{f}(h+x)$),

(S_2) $\forall \theta \in \Theta$, $\forall h \in H$, for μ-almost every ω,

$$\lim_{n\to\infty} l_\theta(\tilde{f}^\theta_h - f^\theta_{(h,\omega)_n})(x) = 0 \text{ for all } x \text{ in } K_\theta.$$

The function will be said to be *weakly s-regular* if the same properties hold, replacing "C^2-function" by "continuous function" and "l_θ" by "m_θ".

The function $\tilde{f}$ is then called the *skeleton* of f.

Remark In properties (S_1) and (S_2), the convergence for all x in K_θ can be replaced by the convergence for $x = 0$. This follows easily from the quasi-invariance of μ and from the definition of m_θ. Likewise, in the definition of m_θ, one can replace "$|t|_H \leq 1$" by "$|t|_H \leq r$", where r denotes any positive fixed real number.

Theorem 4 *Assume that f is s-regular (resp. g is weakly s-regular) with skeleton $\tilde{f}$ (resp. $\tilde{g}$) and let $\xi \in \mathbb{R}^d$. Then $\mathbf{p}_g(\xi) > 0$ if and only if there exists $h \in H$ such that*

$$\tilde{f}(h) = \xi, \ \tilde{g}(h) > 0 \text{ and } \mathrm{rank} D\tilde{f}(h) = d$$

(where $D\tilde{f}$ denotes the differential of $\tilde{f}$).

Lemma 4.1 *Let φ and φ_n be C^2-functions from $\mathbb{R}^d$ into $\mathbb{R}^d$. Let ψ and ψ_n be continuous functions from $\mathbb{R}^d$ into $\mathbb{R}$. Assume that, for any $x \in \mathbb{R}^d$,*

$$\lim_{n\to\infty} \sup(|\varphi - \varphi_n|(x), |D\varphi - D\varphi_n|(x), \sup_{|t|\leq 1} |D^2\varphi - D^2\varphi_n|(x+t)) = 0$$

$$\text{and } \lim_{n\to\infty} \sup_{|t|\leq 1} |\psi - \psi_n|(x+t) = 0.$$

Then, $\varphi(\{J\varphi > 0\} \cap \{\psi > 0\}) \subset \cup_n \varphi_n(\{J\varphi_n > 0\} \cap \{\psi_n > 0\})$.

Proof: Let $\xi = \varphi(x)$ with $J\varphi(x) > 0$ and $\psi(x) > 0$. Then, by the inverse mapping theorem, there exist δ and $\varepsilon > 0$ such that, for n big enough, φ_n is a C^2-diffeomorphism from $B(x,\delta)$ onto an open set containing $B(\varphi_n(x),\varepsilon)$. It can also be assumed that $\psi_n > 0$ on $B(x,\delta)$. There exists n such that $\xi \in B(\varphi_n(x),\varepsilon)$, which yields the result. 2

Suppose then $\mathbf{p}_g(\xi) > 0$. By theorem 3, there exists $\theta \in \Theta$ such that $\mu(\{\omega;\ \exists x \in K_\theta \quad f^\theta_\omega(x) = \xi, g^\theta_\omega(x) > 0 \text{ and } J(f^\theta_\omega)(x) > 0\}) > 0$. By property (S_1) and the previous lemma, $\mu(\{\omega;\ \exists n\ \exists x \in K_\theta \quad \tilde{f}^\theta_{\omega_n}(x) =$

ξ, $\tilde{g}^{\theta}_{\omega_n}(x) > 0$ and $J(\tilde{f}^{\theta}_{\omega_n})(x) > 0\}) > 0$. Consequently, there exist $h \in H$ and $x \in K_\theta$ such that

$$\tilde{f}(h+x) = \xi,\ \tilde{g}(h+x) > 0 \text{ and } \det(\partial_{h_j}\tilde{f}_i)_{1\le i\le d, j\in\theta}(h+x) \neq 0.$$

In particular, $\tilde{f}(h+x) = \xi$, $\tilde{g}(h+x) > 0$ and $\mathrm{rank}D\tilde{f}(h+x) = d$.

Suppose, conversely, that there exists $h^0 \in H$ such that $\tilde{f}(h^0) = \xi$, $\tilde{g}(h^0) > 0$ and $\mathrm{rank}D\tilde{f}(h^0) = d$. Then there exists $\theta \in \Theta$ such that $\tilde{f}^{\theta}_{h^0}(0) = \xi$, $\tilde{g}^{\theta}_{h^0}(0) > 0$ and $J(\tilde{f}^{\theta}_{h^0})(0) > 0$. By property (S_2) and lemma 4.1, for μ-almost every ω, there exists n such that $\xi \in f^{\theta}_{(h^0,\omega)_n}(\{J(f^{\theta}_{(h^0,\omega)_n}) > 0\} \cap \{g^{\theta}_{(h^0,\omega)_n} > 0\})$. Consequently, there exists n such that, if we denote by A the set $\{\omega \in E;\ \xi \in f^{\theta}_{(h^0,\omega)_n}(\{J(f^{\theta}_{(h^0,\omega)_n}) > 0\} \cap \{g^{\theta}_{(h^0,\omega)_n} > 0\})\}$, then $\mu(A) > 0$. Obviously, it can be assumed that $H_n \supset K_\theta$. For μ-almost every ω, the map $h \longrightarrow f^{\theta}_{(h,\omega)_n}$ is continuous from H into $\mathcal{C}^2(K_\theta)$ and the map $h \longrightarrow g^{\theta}_{(h,\omega)_n}$ is continuous from H into $\mathcal{C}(K_\theta)$. Therefore, by the same proof as in lemma 4.1, for all $\omega \in A$, there exists an open set O_ω in H containing h^0, so that

$$h \in O_\omega \Longrightarrow \xi \in f^{\theta}_{(h,\omega)_n}(\{J(f^{\theta}_{(h,\omega)_n}) > 0\} \cap \{g^{\theta}_{(h,\omega)_n} > 0\}).$$

The choice of O_ω may be done in a measurable way with respect to ω. Let then ν_n be the standard Gaussian measure on H_n. As ν_n charges all non-empty open sets and since π_n is an open map from H onto H_n, we have

$$\mu[\{\omega;\ \xi \in f^{\theta}_{\omega}(\{J(f^{\theta}_{\omega}) > 0\} \cap \{g^{\theta}_{\omega} > 0\})\}]$$

$$= \int_{H_n} \mu[\{\omega;\ \xi \in f^{\theta}_{h+\omega'_n}(\{J(f^{\theta}_{h+\omega'_n}) > 0\} \cap \{g^{\theta}_{h+\omega'_n} > 0\})\}]\, d\nu_n(h)$$

$$\ge \int_A \nu_n(\pi_n(O_\omega))\, d\mu(\omega) > 0.$$

Theorem 3 then yields $\mathbf{p}_g(\xi) > 0$.

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