Theory of capacity on the Wiener space

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This text consists of four parts.

In the first one, we develop a fairly general potential theory related to a kernel. Such notions as capacity, equilibrium potentials and equilibrium measures, are studied.
In the second part, we consider the particular setting of the Wiener space and we specially study the capacities $c_{r,p}$ appearing in the Malliavin calculus ([23]).
In the third part, we introduce the notion of a symmetric n-parameter Markov process and we show that, for such processes, hitting probabilities may be estimated in terms of capacities related to an L^2-potential theory. This is applied to give a probabilistic interpretation of capacities $c_{r,2}$ on the Wiener space.
In the last part, we introduce, in the general context described in the first part, "Sobolev spaces" of Banach-valued functions and we use them in the so-called quasi-sure analysis. Here again, the case of Wiener space is specially considered.

1 Analytic potential theory

In this first part, we study a general potential theory from an analytic point of view. Such a theory was developped by H. Sugita ([32]) in the framework of abstract Wiener spaces, with specific methods, and then generalized by T. Kazumi and I. Shigekawa ([21]) to the case of an arbitrary separable metric space equipped with a probability measure and a Markovian semi-group (under some additional assumptions). The theory presented here is more general (since we do not assume the existence of a Markovian semi-group), and the methods are different. In fact, in a recent joint paper with S. Song ([18]), we developped an even more general theory where, in particular, the basic measure is only assumed to be σ-finite. This is important in many examples, even the most classical as the potential theory related to the classical Sobolev spaces in $\mathbb{R}^d$. Here, because of the main example that we shall consider (namely the case of the Wiener space), we shall restrict ourselves to finite measures. The methods are close to that of [15].

Hypotheses

We assume in what follows the following hypotheses:

(H_1) E is a metric space and m is a Borel probability measure on E.

(H_2) $U(x, dy)$ is a Borel kernel on E which satisfies:

$$U1 = 1, \quad mU = m, \quad U(\mathcal{C}_b) \subset \mathcal{C}_b.$$

These hypotheses are far to be minimal (see [18]), but they are simple and often satisfied.

Notation 1.1 Henceforth, we fix a real p, $1 < p < \infty$, and we denote by q the conjugate exponent of p. The symbol $\| \quad \|_p$ will denote the $L^p(m)$-norm. In what follows, L^p is set in place of $L^p(m)$. Kernel U defines a contraction of L^p denoted by $\mathbb{U}_p$.

We also assume:

(H_3) $\mathbb{U}_p$ is injective on L^p.

(Actually, this hypothesis is not necessary, because it is possible to use quotient spaces.)

Notation 1.2 We define space H_p as the image $\mathbb{U}(L^p)$ equipped with the norm $\|\mathbb{U}_p f\|_{H_p} = \|f\|_p$.

Hence, H_p is isometric to L^p and therefore it is a uniformly convex Banach space. If $p = 2$, H_2 is a Hilbert space. Space H_p has to be viewed as a "Sobolev space" defined as a space of "Bessel potentials", which is the classical situation.

Capacity

Notation 1.3 Following an old idea due to D. Feyel ([6]), we define a *functional capacity* γ_p by:
$\forall u$ l.s.c., $u \geq 0$, $\gamma_p(u) = \inf\{\|v\|_{H_p};\, v \in H_p$ and $v \geq u$ m-a.s.$\}$
($\gamma_p(u) = +\infty$ if the above set is empty)
$\forall u :\ E \longrightarrow \overline{\mathbb{R}}$, $\gamma_p(u) = \inf\{\gamma_p(v);\, v$ l.s.c. and $v \geq |u|\}$.

Associated with γ_p, the *capacity* c_p is defined by

$$\forall A \subset E \quad c_p(A) = \gamma_p(1_A).$$

It is easy to see that c_p can also be defined directly, as usually, by:
$\forall O$ open set, $c_p(O) = \inf\{\|u\|_{H_p};\, u \geq 1$ m-a.s. on $O\}$, and
$\forall A \subset E$, $c_p(A) = \inf\{c_p(O);\, O$ open and $O \supset A\}$.
Clearly, if $u \in L^p$, then $\gamma_p(u) \geq \|u\|_p$. In particular, if A is a Borel set, $c_p(A) \geq (m(A))^{1/p}$.

In the rest of this section, we generally omit, for simplicity, p in the notation.

Let us give some direct consequences of the definitions:

(*i*) $\forall f\ \gamma(f) = \gamma(|f|)$ and $\gamma(1) = 1$.

(*ii*) $\forall f, g \geq 0\ \gamma(f+g) \leq \gamma(f) + \gamma(g)$ and $f \leq g \Longrightarrow \gamma(f) \leq \gamma(g)$,
$\forall f\ \forall \lambda \in \mathbb{R}\ \gamma(\lambda f) = |\lambda|\gamma(f)$.

(*iii*) For any non negative l.s.c. function f such that $\gamma(f) < \infty$, there exists a unique $\varphi \in H$ such that $\varphi \geq f$ m-a.s. and $\|\varphi\|_H = \gamma(f)$. (This is a consequence of the projection theorem in uniformly convex spaces.)

(*iv*) Let $(f_n)_{n\geq 0}$ be an increasing sequence of non negative l.s.c. functions, then $\gamma(\lim_{n\to\infty} f_n) = \lim_{n\to\infty} \gamma(f_n)$.

We then easily obtain:

Proposition 1.1 *For any sequence (f_n) of functions*

$$\gamma(\sum_n |f_n|) \leq \sum_n \gamma(f_n) \quad (\sigma\text{-subadditivity}).$$

For any sequence (A_n) of subsets of E

$$c(\cup_n A_n) \leq \sum_n c(A_n).$$

Let us now give some basic definitions.

Definitions 1.1 • A *polar set* is a set A such that $c(A) = 0$. In particular, a Borel polar set is m-negligible and, by σ-subadditivity, a countable union of polar sets is polar.

- A property is said to hold *quasi-everywhere* (q.e.) if it holds out of a polar set.
- A *nest* is an increasing sequence (F_k) of closed sets in E such that $c(F_k^c) \to 0$ when $k \to \infty$. (Superscript c denotes the complement.)
- A *quasi-continuous function* is a function f on E such that there exists a nest (F_k) so that, for any k, f is finite continuous on F_k (Lusin's property with respect to the capacity).

It is easy to obtain from the definitions the following basic facts:

(v) $\forall f \quad \gamma(f) = 0 \Longleftrightarrow f = 0$ q.e.

(vi) $\forall f \quad \gamma(f) < +\infty \Longrightarrow f$ is finite q.e.

(vii) $\forall f, g \quad |f| \leq |g|$ q.e. $\Longrightarrow \gamma(f) \leq \gamma(g)$.

An important property is then the following.

Proposition 1.2 *Let f be a real quasi-continuous function and let O be an open set. Then,*

$$f \geq 0 \;\; m\text{-}a.s. \;\; on \; O \Longrightarrow f \geq 0 \;\; q.e. \;\; on \; O.$$

Proof: Let $N = O \cap \{f < 0\}$ and let (F_k) be a nest such that f is continuous on each F_k. The set $O_k = \{f < 0\} \cup F_k^c$ is an open set and we have $O_k \cap O = N \cup (F_k^c \cap O)$. Hence, by the definition of the capacity of open sets, as N is m-negligible, $c(O_k \cap O) = c(F_K^c \cap O)$. Finally, $c(N) \leq c(O_k \cap O) \leq c(F_k^c)$, and therefore, $c(N) = 0$. 2

Space $L^1(\gamma)$

Notation 1.4 Again following D. Feyel [6], we define:

$$\begin{array}{rcl} \mathcal{F}^1(\gamma) & = & \{f : \; E \longrightarrow \overline{\mathbb{R}}; \;\; f \text{ q.e. defined and } \gamma(f) < +\infty\} \\ \mathcal{L}^1(\gamma) & = & \{f \in \mathcal{F}^1(\gamma); \;\; \exists(\varphi_n) \subset \mathcal{C}_b \;\; \gamma(f - \varphi_n) \to 0\} \\ L^1(\gamma) & = & \text{Quotient space of } \mathcal{L}^1(\gamma) \text{ by the relation} \\ & & \text{of equality quasi-everywhere.} \end{array}$$

Then, $L^1(\gamma)$ is a vector space that we equip with the norm $\|f\|_\gamma = \gamma(f)$.

Proposition 1.3 *1. Each element of $\mathcal{L}^1(\gamma)$ is quasi-continuous.*

2. $L^1(\gamma)$ is a Banach lattice.

The proof is not difficult. The following theorem relates space H with space $L^1(\gamma)$.

Theorem 1.1 *1. Each element $h \in H$ admits a quasi-continuous m-representative $\tilde{h}$ which is unique up to quasi-everywhere equality.*

2. $\forall h \in H$, $\tilde{h} \in \mathcal{L}^1(\gamma)$ and $\gamma(\tilde{h}) \leq \|h\|_H$.

3. If $h = \mathbb{U}g \in H$ with $g \in L^p$, then for any Borel m-representative g_0 of g, Ug_0 is defined q.e. and $\tilde{h} = Ug_0$ q.e.

Proof: First of all, the uniqueness of the quasi-continuous representative follows from proposition 1.2. Let $g \in L^p$. If ψ is an l.s.c. function such that $\psi \geq |g_0|$,

$$\gamma(U|g_0|) \leq \gamma(U\psi) \leq \|\psi\|_p$$

by definition of γ and the fact that, by (H_2), $U\psi$ also is l.s.c. By exterior regularity of m (E is metric), we then have $\gamma(U|g_0|) \leq \|g\|_p$. In particular, $U|g_0|$ is finite q.e. and then Ug_0 is defined q.e. and $\gamma(Ug_0) \leq \|g\|_p$. Let (φ_n) be a sequence in $\mathcal{C}_b$ converging to g in L^p. By what precedes, $\gamma(Ug_0 - U\varphi_n) \leq \|g - \varphi_n\|_p$ and therefore, by (H_2), $Ug_0 \in \mathcal{L}^1(\gamma)$. 2

Remark 1.1 By identification of h and the class of $\tilde{h}$ in $\mathcal{L}^1(\gamma)$, space H may be considered as a subspace of $L^1(\gamma)$ (with a finer norm). We shall often do this identification. Then, space $L^1(\gamma)$ is a Banach *lattice* which contains space H. This is one of the main interests of $L^1(\gamma)$ because, in general, H is not a lattice.

By a similar proof to that of theorem 1.1, we obtain the following result.

Proposition 1.4 *Let V be a Borel kernel satisfying the same hypothesis (H_2) as U. Assume that its extension $\mathbb{V}_p$ to L^p satisfies $\mathbb{V}_p(L^p) \subset H$. Then, for any $g \in L^p$, for any g_0 Borel m-representative of g, Vg_0 is defined q.e. and belongs to $\mathcal{L}^1(\gamma)$. In particular, Vg_0 is a quasi-continuous representative of $\mathbb{V}_p g$.*

We now have the following useful characterization of elements of $\mathcal{L}^1(\gamma)$.

Proposition 1.5 *For a function f, the following properties are equivalent:*

i) $f \in \mathcal{L}^1(\gamma)$.

ii) $f \in \mathcal{F}^1(\gamma)$ *and* f *is quasi-continuous.*

iii) f *is quasi-continuous and* $\exists h \in H$ *such that* $|f| \leq h$ *a.s.*

Proof: By proposition 1.3, $i) \Longrightarrow ii)$, and, by definition of $\mathcal{F}^1(\gamma)$, $ii) \Longrightarrow iii)$.
Suppose now that f is a bounded quasi-continuous function. Let (F_k) be a nest such that, for any k, f is continuous on F_k. By Tietze's theorem, there exists $g_k \in \mathcal{C}_b$ such that $g_k|_{F_k} = f|_{F_k}$ and $\|g_k\|_\infty \leq \|f\|_\infty$. Then $\gamma(f - g_k) \leq 2\|f\|_\infty c(F_k^c)$ which tends to 0 when k tends to infinity. Therefore $f \in \mathcal{L}^1(\gamma)$.
Suppose finally that property $iii)$ holds. Let g_0 be a Borel representative of $g \in L^p$ such that $h = \mathbb{U}g$. Then, $|f| \leq Ug_0$ q.e. (by proposition 1.2 and theorem 1.1). Denoting in what follows sup (resp. inf) by $\vee$ (resp. $\wedge$) and setting $f_n = \frac{nf}{n \vee |f|}$, by what precedes $f_n \in \mathcal{L}^1(\gamma)$ and, on the other hand,

$$|f - f_n| \leq \tilde{h} - \tilde{h} \wedge n \leq U(g_0 - g_0 \wedge n) \text{ q.e.}$$

Therefore, by theorem 1.1, $\gamma(f - f_n) \leq \|g - g \wedge n\|_p$ which tends to 0 when n tends to infinity. This implies that property $i)$ holds. 2

Potentials

The following proposition shows that, for any $g \in \mathcal{F}^1(\gamma)$, $\gamma(g)$ may be realized as a minimum.

Proposition 1.6 *Let* $g \in \mathcal{F}^1(\gamma)$. *Then*

$$\gamma(g) = \min\{\|h\|_H; \ \ \tilde{h} \geq |g| \ \ q.e.\}.$$

This minimum is achieved by a unique $\phi_g \in H$ *which is called the* equilibrium potential *of* g.

Proof: By the projection theorem in uniformly convex spaces and the fact that, by theorem 1.1, $\{h \in H; \ \tilde{h} \geq |g| \text{ q.e.}\}$ is a closed (convex) subset of H, the minimum is achieved by a unique $\phi_g \in H$. We have $\widetilde{\phi_g} \geq |g|$ q.e. and therefore, again by theorem 1.1, $\gamma(g) \leq \|\phi_g\|_H$. Let now λ satisfy $\gamma(g) < \lambda$. There exists an l.s.c. function φ such that $|g| \leq \varphi$ and $\gamma(\varphi) < \lambda$. There exists $h \in H$ such that $\|h\|_H < \lambda$ and $h \geq \varphi$ m-a.s. We then have, by proposition 1.2, $\tilde{h} \geq \varphi$ q.e. and therefore $\tilde{h} \geq |g|$ q.e. Hence $\|\phi_g\|_H \leq \|h\|_H < \lambda$ and finally $\|\phi_g\|_H \leq \gamma(g)$. 2
An important corollary is the following.

Corollary 1.6.1 *For any increasing sequence (f_n) of non negative functions,*

$$\gamma(\lim_n f_n) = \lim_n \gamma(f_n).$$

In particular, if (A_n) is an increasing sequence of subsets of E,

$$c(\cup_n A_n) = \lim_n c(A_n).$$

Notation 1.5 We denote by $\mathcal{U}$ the set of non negative finite u.s.c. functions on E with compact support.

Concerning decreasing sequences, it is clear that, if (φ_n) is a decreasing sequence in $\mathcal{U}$, then $\gamma(\lim_n \varphi_n) = \lim_n \gamma(\varphi_n)$. Likewise, if (K_n) is a decreasing sequence of compact sets, $c(\cap_n K_n) = \lim_n c(K_n)$. As a consequence of these properties of the capacity with respect to monotone sequences, γ is a so-called *functional Choquet capacity* and c is a *Choquet capacity*, which implies that Choquet's capacitability theorem is valid.
We now introduce the general definition of a potential.

Definition 1.2 A *potential* is an element $u \in H$ which satisfies:

$$\forall v \in H \quad v \geq u \Longrightarrow \|v\|_H \geq \|u\|_H.$$

Of course, equilibrium potentials are potentials: Namely, if $v \in H$ and $v \geq \phi_g$, then $\tilde{v} \geq \widetilde{\phi_g} \geq |g|$ q.e., and therefore, $\|v\|_H \geq \|\phi_g\|_H$. The converse will follow from the following theorem which shows that any potential is the equilibrium potential of itself. Let us first introduce further notation.

Notation 1.6
$\mathcal{P}$: The set of potentials.
T: The isometry from L^p onto H given by $\mathbb{U}$.
$S = T^{-1}$.
T^* (resp. S^*) denotes the adjoint of T (resp. S).
i denotes the canonical embedding from H into L^p. Consequently, $\mathbb{U} = i \circ T$. The adjoint of i is denoted by i^*.
For any set F of (classes of) functions, F^+ denotes the set of non negative elements of F. If F is a normed space, we denote by F' (resp. F^*) the set of linear continuous functionals (resp. linear functionals) on F, and by F'^+ (resp. F^{*+}) the subspace consisting of those functionals which are non negative on F^+. It can be noticed that $T((L^p)^+) \subset H^+$ and therefore $T^*(H'^+) \subset (L^q)^+$.

We have the following characterization which establishes a one-to-one correspondence between $\mathcal{P}$ and H'^+.

Theorem 1.2 *Let $u \in H$. The following statements are equivalent:*

(1) $u \in \mathcal{P}$

(2) $Su \geq 0$ *and* $S^*((Su)^{p/q}) \in H'^+$

(3) $\exists \nu \in H'^+ \quad u = T((T^*\nu)^{q/p})$

(4) $u \geq 0$ *and* $\gamma(\tilde{u}) = \|u\|_H$

(5) $u = \phi_u$

If $u \in \mathcal{P}$, then ν given by (3) *is unique. Potential u is then called the* potential generated *by ν and it is denoted by u_ν.*

Proof: Let $u \in \mathcal{P}$. For any $v \in H^+$ and for any $t \geq 0$, $\|u + tv\|_H \geq \|u\|_H$. Then,

$$\left(\frac{d^+}{dt}\|u+tv\|_H^p\right)_{t=0} = p\int |Su|^{p-1}\mathrm{sign}(Su)\ Sv\ dm \geq 0.$$

Consequently, $Su \geq 0$ and $S^*((Su)^{p-1}) \in H'^+$. Thus (1) $\Longrightarrow$ (2).

If (2) holds, by what precedes and an argument of convexity, for any $v \in H^+$ and for any $t \geq 0$, $\|u+tv\|_H \geq \|u\|_H$ and (1) follows.

Clearly, (2) $\Longleftrightarrow$ (3) and ν is unique, given by $S^*((Su)^{p/q})$.

Suppose $u \in \mathcal{P}$. Then $Su \geq 0$ and hence $u = TSu \geq 0$. By definition of $\mathcal{P}$, $\min\{\|v\|_H;\ \ \tilde{v} \geq \tilde{u}\}$ is achieved by u and therefore (4) holds.

By uniqueness of the equilibrium potential, (4) $\Longrightarrow$ (5), and, since equilibrium potentials are potentials, (5) $\Longrightarrow$ (1). 2

Remark 1.2 If $p = 2$, the map $\nu \in H'^+ \longrightarrow u_\nu \in \mathcal{P}$ given in the previous theorem is the restriction to H'^+ of the *linear isometry* TT^* from H' onto H, which is the Riesz isometry from H' onto H. Namely, for any $\varphi \in H$,

$$< \varphi, \nu >_{H,H'} = (S\varphi, Su_\nu)_{L^2} = (\varphi, u_\nu)_H.$$

The following density property is an easy consequence of the Hahn-Banach theorem.

Proposition 1.7 *The set $i^*((L^q)^+)$ is dense in H'^+ (with respect to the metric defined by the norm of H').*

The following corollary could be used for defining $\mathcal{P}$ in the Hilbert case.

Corollary 1.7.1 *In the case $p = 2$, the set $\mathbb{UU}^*((L^2)^+)$ is dense in $\mathcal{P}$ (with respect to the metric defined by the norm of H).*

We now study the dual of $L^1(\gamma)$.

Proposition 1.8

$$L^1(\gamma)' = L^1(\gamma)^{*+} - L^1(\gamma)^{*+}$$

The proof is standard. In particular, $L^1(\gamma)^{*+} = L^1(\gamma)'^{+}$.

Theorem 1.3 *For any $L \in L^1(\gamma)'^{+}$, denote by ν_L the "restriction" of L to H: $h \in H \longrightarrow < \tilde{h}, L >_{L^1(\gamma),L^1(\gamma)'}$. Then $\nu_L \in H'^{+}$ and $\|\nu_L\|_{H'} = \|L\|_{L^1(\gamma)'}$.*

The map $L \in L^1(\gamma)^{*+} \longrightarrow \nu_L \in H'^{+}$ is surjective and $H^{*+} = H'^{+}$.

Proof: Let $L \in L^1(\gamma)'^{+}$. By theorem 1.1, $\nu_L \in H'^{+}$ and $\|\nu_L\|_{H'} \leq \|L\|_{L^1(\gamma)'}$. Let $u \in L^1(\gamma)$. Then $|u| \leq \widetilde{\phi_u}$ q.e. and hence

$$| < u, L >_{L^1(\gamma),L^1(\gamma)'} | \leq < \widetilde{\phi_u}, L >_{L^1(\gamma),L^1(\gamma)'} =$$

$$< \phi_u, \nu_L >_{H,H'} \leq \|\nu_L\|_{H'} \|\phi_u\|_H = \|\nu_L\|_{H'} \gamma(u).$$

Therefore $\|L\|_{L^1(\gamma)'} \leq \|\nu_L\|_{H'}$. Finally, since for any $u \in \mathcal{L}^1(\gamma)$ there exists $h \in H$ such that $|u| \leq \tilde{h}$ q.e., by a classical consequence of the Hahn-Banach theorem, for any $\nu \in H^{*+}$, map ν extends into $L \in L^1(\gamma)^{*+}$. Then $\nu = \nu_L \in H'^{+}$. 2

Remark 1.3 If we assume that H is dense in $L^1(\gamma)$, of course the above map $L \longrightarrow \nu_L$ is a one-to-one correspondence between $L^1(\gamma)'^{+}$ and H'^{+}, preserving the norms.

Finite energy measures

From now on, we assume a fourth hypothesis:

(H_4) (Tightness of capacity c) There exists a nest (K_k) consisting of compact sets.

Proposition 1.9 *For any non negative Borel function f,*

$$\gamma(f) = \sup\{\gamma(\varphi);\ \ \varphi \in \mathcal{U} \text{ and } \varphi \leq f\}$$

(where $\mathcal{U}$ was defined in notation 1.5). Likewise, for any Borel subset A of E,

$$c(A) = \sup\{c(K);\ \ K \text{ compact and } K \subset A\}.$$

This is a consequence of Choquet's capacitability theorem because, thanks to (H_4), Borel sets are analytic up to a polar set. Another consequence of hypothesis (H_4) is the following.

Proposition 1.10 *Space $L^1(\gamma)$ is a separable Banach space.*

Proof: Let D be a countable set in $\mathcal{C}_b$, dense for the topology of uniform convergence on the sets K_k, $k \geq 0$. Then $\{\varphi 1_{K_k};\ \ \varphi \in D,\ k \geq 0\} = \mathcal{D}$ is countable and $L^1(\gamma)$is contained in the closure of $\mathcal{D}$ in $\mathcal{F}^1(\gamma)$. 2

Definition 1.3 The σ-algebra generated by Borel sets and polar sets will be called the σ-algebra of *quasi-Borel sets*.

It is clear that a set A is quasi-Borel iff there exist B_1 and B_2 Borel sets such that $B_1 \subset A \subset B_2$ and $c(B_2 \backslash B_1) = 0$. The σ-algebra of quasi-Borel sets also is the σ-algebra generated by $\mathcal{L}^1(\gamma)$. We have then the following representation theorem.

Theorem 1.4 *For any $L \in L^1(\gamma)'^+$, there exists a unique measure on the σ-algebra of quasi-Borel sets, l, such that*

$$\mathcal{L}^1(\gamma) \subset \mathcal{L}^1(l) \text{ and } \forall f \in \mathcal{L}^1(\gamma) \quad < f, L >_{L^1(\gamma), L^1(\gamma)'} = \int f \, dl,$$

where $\mathcal{L}^1(l)$ denotes the set of l-integrable functions. Moreover, for any quasi-Borel set A,

$$l(A) \leq \|L\|_{L^1(\gamma)'} c(A)$$

(and, in particular, l does not charge polar sets).

Proof: By Daniell's theorem, we have essentially to prove that, if (φ_n) is a decreasing sequence in $\mathcal{L}^1(\gamma)$ pointwise converging to 0, then $\lim_n \gamma(\varphi_n) = 0$. Let (φ_n) be such a sequence. Let (A_k) be a nest such that, for any n and k, φ_n is continuous on A_k (such a nest exists). By Dini's lemma, for all k, m, $\lim_{n\to\infty} \varphi_n 1_{K_m \cap A_k} = 0$ uniformly. Therefore, for any $N > 0$,

$$\limsup_{n\to\infty} \gamma(\varphi_n) \leq \gamma(\varphi_1 1_{(K_m \cap A_k)^c}) \leq N(c(K_m^c) + c(A_k^c)) + \gamma(\varphi_1 - \varphi_1 \wedge N).$$

There exists $h \in H$ such that $\varphi_1 \leq \tilde{h}$ q.e. There exists $g \in L^p$ such that $h = \mathbb{U}g$. By the same argument as in the proof of proposition 1.5, $\gamma(\varphi_1 - \varphi_1 \wedge N) \leq \|g - g \wedge N\|_p$. It then suffices to let m, k and N tend to infinity.
In particular, if $\varphi \in \mathcal{C}_b^+$, then $\int \varphi \, dl \leq \|L\| \gamma(\varphi)$. By increasing limit, for any open set O, $l(O) \leq \|L\| c(O)$ and then, by definition of the capacity, for any quasi-Borel set A, $l(A) \leq \|L\| c(A)$. 2
This leads to the following definition.

Definition 1.4 The measures l appearing in the previous theorem are called *finite energy measures.*

Notation 1.7 We shall denote by $\mathcal{E}$ the set of finite energy measures.

Clearly $\nu \in \mathcal{E}$ iff ν is a measure on the quasi-Borel σ-algebra such that

$$\exists C \geq 0 \ \ \forall f \in \mathcal{L}^1(\gamma)^+ \quad \int f \, d\nu \leq C\gamma(f).$$

In particular, a quasi-Borel measure which is dominated by a finite energy measure is a finite energy measure. By theorems 1.3 and 1.4, we obtain the following correspondence between H'^+ and $\mathcal{E}$.

Proposition 1.11 *For any $\nu \in H'^+$, there exists $l \in \mathcal{E}$ such that*

$$\forall h \in H \quad < h, \nu >_{H,H'} = \int \tilde{h} \, dl.$$

Conversely, any element l in $\mathcal{E}$ thus defines a unique $\nu \in H'^+$.

As a consequence we obtain:

Corollary 1.11.1 *Any finite energy measure l defines, according to the previous proposition, an element ν of H'^+ which generates, by theorem 1.2, a potential u. We shall also say that u is the* potential generated *by l and we shall use the notation $u = u_l$. Conversely, any potential u is generated by a finite energy measure l. We then have*

$$\forall h \in H \quad \int \tilde{h} \, dl = \int (Sh)(Su_l)^{p/q} \, dm \ \ (= (h, u_l)_H \text{ if } p = 2).$$

In particular, $\int \widetilde{u_l} \, dl = \|u_l\|_H^p$.

Notation 1.8 We denote by (H_5) the following hypothesis:

(H_5) Space H is dense in $L^1(\gamma)$.

Clearly, if (H_5) is satisfied, the finite energy measure generating a given potential is uniquely determined by this potential and there are bijective correspondences between $(L^1(\gamma))'^+$, H'^+, $\mathcal{E}$ and $\mathcal{P}$.

Equilibrium measures

We assume hypotheses (H_1) to (H_4).

Definitions 1.5

- Let $g \in \mathcal{F}^1(\gamma)$. Any finite energy measure generating the equilibrium potential ϕ_g is called an *equilibrium measure* of g. (If (H_5) also is satisfied, this measure is uniquely determined.)
- A *quasi-upper semicontinuous* (q.u.s.c.) function is a function g such that there exists a decreasing sequence (g_n) in $\mathcal{L}^1(\gamma)$ such that $g = \lim_n g_n$ q.e.

The main result is the following.

Theorem 1.5 *Let $g \in \mathcal{F}^1(\gamma)^+$ be a q.u.s.c. function. Then there is an equilibrium measure of g, denoted by ν_g, which is carried by $\{g > 0\} \cap \{g = \widetilde{\phi_g}\}$. In particular*

$$\gamma(g)^p = \int g \, d\nu_g.$$

Proof: The following proof was suggested by D. Feyel.

We identify, by theorem 1.4, $\mathcal{E}$ with $L^1(\gamma)'^+$ and we denote by $\mathcal{K}$ the set $\{\nu \in \mathcal{E};\ \|\nu\|_{L^1(\gamma)'} \leq 1\}$ equipped with the weak topology $\sigma(L^1(\gamma)', L^1(\gamma))$ for which it is compact. Let $g \in \mathcal{F}^1(\gamma)^+$ be a q.u.s.c. function and let (g_n) be a corresponding decreasing sequence in $\mathcal{L}^1(\gamma)$. By the Hahn-Banach theorem, for any n there exists a linear functional L_n on $L^1(\gamma)$ such that $L_n(g_n) = \gamma(g_n)$ and, for any $\varphi \in L^1(\gamma)$, $L_n(\varphi) \leq \gamma(\varphi^+)$. Clearly $L_n \in L^1(\gamma)'^+$ and $\|L_n\|_{L^1(\gamma)'} \leq 1$. Therefore there exists $\nu_n \in \mathcal{K}$ such that $\int g_n \, d\nu_n = \gamma(g_n)$. Hence $\gamma(g_n) = \max_{\nu \in \mathcal{K}} \int g_n \, d\nu$. By a classical min-max theorem, since $\mathcal{K}$ is compact, we then have

$$\gamma(g) \geq \max_{\nu \in \mathcal{K}} \int g \, d\nu = \max_{\nu \in \mathcal{K}} \inf_n \int g_n \, d\nu = \inf_n \max_{\nu \in \mathcal{K}} \int g_n \, d\nu = \inf_n \gamma(g_n) \geq \gamma(g).$$

Consequently, there exists $\nu \in \mathcal{K}$ such that $\gamma(g) = \int g \, d\nu$. Replacing ν by $1_{\{g>0\}}\nu$, we may assume that ν is carried by $\{g > 0\}$. We may also assume that $\gamma(g) \neq 0$. Let us still denote by ν the element of H'^+ associated with ν by proposition 1.11. By theorem 1.3, $\|\nu\|_{H'} = \|\nu\|_{L^1(\gamma)'} = 1$ and therefore $\gamma(g) = \int g \, d\nu \leq \int \widetilde{\phi_g} \, d\nu =< \phi_g, \nu >_{H,H'} =< S\phi_g, T^*\nu >_{L^q,L^p} \leq \|\phi_g\|_H \|\nu\|_{H'} = \gamma(g)$. It follows that ν is carried by $\{g = \widetilde{\phi_g}\}$ and, by the case of equality in Hölder's inequality, $S\phi_g = \gamma(g)[T^*\nu]^{q/p}$. Therefore we can set $\nu_g = (\gamma(g))^{p/q}\nu$. 2

Corollary 1.5.1 *Let F be a closed set. Denote by ϕ_F the equilibrium potential of 1_F and by ν_F a corresponding equilibrium measure as in the previous theorem. Then*

$$\nu_F(F^c) = 0, \quad \widetilde{\phi_F} = 1 \ \ \nu_F\text{-}a.e., \quad c(F)^p = \int d\nu_F.$$

As a consequence, we have the following dual characterization of polar sets.

Corollary 1.5.2 *Let A be a quasi-Borel set. Then A is polar if and only if*

$$\forall \nu \in \mathcal{E} \quad \nu(A) = 0.$$

Proof: The necessity has been shown in theorem 1.4. Conversely, if A is not polar, by proposition 1.9 there exists a compact subset K of A which is not polar. Then $\nu_K(A) \geq \nu_K(K) = c(K)^p > 0$. 2

2 Capacities on Wiener space

We shall now consider the framework of the classical Wiener space (according to [7], we could, more generally, consider the case of a locally convex Lusin space with a centered Gaussian measure). We shall prove that the classical capacities $c_{r,p}$ appearing in Malliavin's calculus are of the type studied in the first section whose hypotheses are satisfied.

Notation 2.1 In this section, E denotes the classical Wiener space $\mathcal{C}_0(\mathbb{R}_+;\mathbb{R}^d)$ equipped with its usual topology and m denotes the Wiener measure on E, considered as a Borel measure. Hence, hypothesis (H_1) is satisfied.

We denote by $(B_t)_{t\geq 0}$ the coordinates process which is, under m, the standard Brownian process in $\mathbb{R}^d$ starting from 0. We denote by B_t^j, $1 \leq j \leq d$, the components of B_t.

For $t \geq 0$, if f is a non negative Borel function on E, we set

$$P_t f(x) = \int f(\mathrm{e}^{-t/2}x + \sqrt{1-\mathrm{e}^{-t}}y)\, dm(y) \quad (\text{ Mehler's formula}).$$

Then P_t is a Borel kernel, $P_t 1 = 1$, $mP_t = m$, $P_t(\mathcal{C}_b) \subset \mathcal{C}_b$ and

$$\forall f, g \geq 0 \quad \int P_t f \; g \, dm = \int f \; P_t g \, dm \qquad \text{(symmetry)},$$

$$\forall t, s \geq 0 \quad P_{t+s} = P_t P_s \qquad \text{(semi-group property)}.$$

The semi-group of Borel kernels $(P_t)_{t\geq 0}$ is called the *Ornstein-Uhlenbeck semi-group.*

We denote, for $1 < p < \infty$, by $(\mathbb{P}_{t,p})_{t\geq 0}$ the $L^p(m)$-extension of $(P_t)_{t\geq 0}$. Then $(\mathbb{P}_{t,p})_{t\geq 0}$ is a strongly continuous sub-Markovian contraction semi-group on $L^p(m)$, and, for $t \geq 0$, $\mathbb{P}_{t,2}$ is symmetric on $L^2(m)$. We denote by A_p the infinitesimal generator of the semi-group $(\mathbb{P}_{t,p})_{t\geq 0}$: Operator A_p is called the *Ornstein-Uhlenbeck operator* in $L^p(m)$.

We set, for any real number $r > 0$,

$$U^r = \frac{1}{\Gamma(r/2)} \int t^{\frac{r}{2}-1} \mathrm{e}^{-t} P_t \, dt.$$

Then U^r satisfies the same properties as P_t. In particular, kernel U^r satisfies (H_2). We denote by $\mathbb{U}_p^r$ the $L^p(m)$-extension of U^r. We then have $\mathbb{U}_p^r = (I - A_p)^{-r/2}$. Consequently, hypothesis (H_3) also is satisfied ($\mathbb{U}_p^r$ is injective). The associated space $H_p = \mathbb{U}_p^r(L^p(m))$ will be denoted by $\mathbb{D}_p^r$. According to the previous section, $\mathbb{D}_p^r$ is equipped with the norm $\|\mathbb{U}_p^r f\|_{r,p} = \|f\|_p$.

The previous potential theory can then be developped for any fixed $r > 0$ and $1 < p < \infty$. The corresponding capacities will be denoted by $\gamma_{r,p}$ and $c_{r,p}$. We shall also use the terminology (r,p)-polar, (r,p)-quasi-continuous, ... in place of $c_{r,p}$-polar, $c_{r,p}$-quasi-continuous, ... Space $\mathbb{D}_p^r$ is decreasing with respect to r and p, while $\gamma_{r,p}$ and $c_{r,p}$ are increasing with respect to r and p. We shall denote by $\mathbb{D}^\infty$ the set $\bigcap_{r>0,p>1} \mathbb{D}_p^r$.

Definition 2.1 A *slim set* is a set which is (r,p)-polar for any $r > 0$ and $1 < p$.

The following proposition comes easily from the definitions.

Proposition 2.1 *If $f \in \mathbb{D}^\infty$, then there exists an m-representative of f, $\tilde{f}$, which belongs to $\bigcap_{r>0,p>1} \mathcal{L}^1(\gamma_{r,p})$ and which is unique up to equality out of a slim set.*

The following result was first proved in [32] by using the differential definition of $\mathbb{D}_p^1$ (Meyer's inequalities). It also is a direct consequence of the holomorphy of $\mathbb{P}_{t,p}$ which follows from the symmetry of $(\mathbb{P}_{t,2})$ (see [31]).

Proposition 2.2

$$\forall r > 0 \;\; \forall p > 1 \;\; \forall t > 0 \quad \mathbb{P}_{t,p}(L^p(m)) \subset \mathbb{D}_p^r$$

Then, by proposition 1.4, we get:

Corollary 2.2.1 *For all $r > 0$, for all $p > 1$, for all Borel function f such that $\int |f|^p \, dm < \infty$, $P_t f$ is defined (r,p)-q.e. and $P_t f \in \mathcal{L}^1(\gamma_{r,p})$.*

As a consequence, we have the following improvement of a classical 0-1 law (cf. [7]).

Proposition 2.3 *Let G be a Borel linear subspace of E. If $m(G) > 0$, then $m(G) = 1$ and, more precisely, the complement G^c is a slim set.*

Proof: We have, by Mehler's formula, $P_t 1_G = m(G)$ on G. Letting t tend to 0, we obtain $1 = m(G)$ a.s. on G. Therefore, if $m(G) > 0$, then $m(G) = 1$. We then have, again by Mehler's formula, $P_t 1_G = 1_G$ for any $t \geq 0$. Hence, by corollary 2.2.1, $1_G \in \mathcal{L}^1(\gamma_{r,p})$. As $1_G = 1$ a.s., $1_G = 1$ (r,p)-q.e. and therefore $c_{r,p}(G^c) = 0$. 2

Example If $0 < \alpha < 1/2$, the set of elements of E which are not locally Hölder continuous of order α is a slim set.

The following result ([7]) has many applications.

Proposition 2.4 *Let $q : E \longrightarrow [0,\infty]$ be a Borel function which is sublinear (i.e. $\forall x, y$ $q(x+y) \leq q(x) + q(y)$, $q(0) = 0$, $\forall \lambda > 0$ $\forall x$ $q(\lambda x) = \lambda q(x)$). If q is finite a.s., then $q \in \bigcap_{r>0,p>1} \mathcal{L}^1(\gamma_{r,p})$.*

Proof: Let $\hat{q}(x) = q(x) + q(-x)$. Then $\hat{q}$ also is finite a.s. Since $Z = \{\hat{q} < \infty\}$ is a Borel linear subspace of E, Z^c is, by proposition 2.3, a slim set. By Fernique's theorem, $\hat{q} \in \cap_{p>1} L^p(m)$. Then, by corollary 2.2.1, $P_t q \in \cap_{r>0,p>1} \mathcal{L}^1(\gamma_{r,p})$. Now, by Mehler's formula and the sublinearity of q, for any $x \in E$,

$$\mathrm{e}^{-t/2} q(x) - \sqrt{1-\mathrm{e}^{-t}} \int q\, dm \leq P_t q(x) \leq \mathrm{e}^{-t/2} q(x) + \sqrt{1-\mathrm{e}^{-t}} \int q\, dm$$

and therefore, $|\mathrm{e}^{t/2} P_t q - q| \leq \sqrt{\mathrm{e}^t - 1} \int q\, dm$ on Z. Consequently, $\mathrm{e}^{t/2} P_t q$ tends to q uniformly on Z and the result follows. 2

Remark Under the assumptions of the proposition, by [7], $q \in \cap_{p>1} \mathbb{D}^1_p$ also holds.

We now give a few corollaries (see [7]).

Corollary 2.4.1 *For any $r > 0$ and $p > 1$, capacity $c_{r,p}$ satisfies tightness property (H_4).*

Proof: Let K be a convex symmetric compact subset of E such that $m(K) > 0$ (such a set obviously exists) and let q be the Minkowski functional associated with K:

$$q(x) = \inf\{\lambda > 0; \ \ x \in \lambda K\} \leq +\infty.$$

Then q is an l.s.c. sublinear symmetric function and $m(q < +\infty) > 0$. Therefore, by propositions 2.3 and 2.4, $q \in \mathcal{L}^1(\gamma_{r,p})$. Set $K_n = \{q \leq n\}$. Then $K_n = nK$ is a compact set and $c_{r,p}(K_n^c) \leq n^{-1} \gamma_{r,p}(q)$ tends to 0 as n tends to ∞. 2

Corollary 2.4.2 *For $1 \leq j \leq d$,*

$$\limsup_{t\to\infty} \frac{B_t^j}{\sqrt{2t \log\log t}} = 1$$

out of a slim set.

Proof: Define q on E by $q(\omega) = \limsup_{t\to\infty} \frac{B_t^j(\omega)}{\sqrt{2t\log\log t}}$. Then q satisfies hypotheses of proposition 2.4. Hence $q \in \mathcal{L}^1(\gamma_{r,p})$ and consequently q is (r,p)-quasi-continuous. As $q = 1$ a.s., $q = 1$ (r,p)-q.e. 2

Corollary 2.4.3 *Let $\mathcal{H}$ be the Cameron-Martin space ($\mathcal{H} = \{\int_0^\cdot \varphi(s)\, ds;\ \ \varphi \in L^2(\mathbb{R}_+;\mathbb{R}^d)\}$). Then $\mathcal{H}$ is a slim set.*

Proof: As, for any $h \in \mathcal{H}$, $\limsup_{t\to\infty} \frac{|h(t)|}{\sqrt{2t\log\log t}} = 0$, it suffices to apply the previous corollary. 2

We now prove that the last assumption (H_5) also holds.

Proposition 2.5 *For any $r > 0$ and $p > 1$, $\mathbb{D}_p^r$ is dense in $L^1(\gamma_{r,p})$, which means that property (H_5) is satisfied.*

Proof: There are many proofs of this fact. Here we use a general method based on the Hahn-Banach theorem. Let $L \in L^1(\gamma)'$ which vanishes on $\mathbb{D}_p^r$. By proposition 1.8, theorem 1.4 and proposition 2.2, there exist l_1 and l_2 (r,p)-finite energy measures such that, for any $t > 0$ and $\varphi \in \mathcal{C}_b$, $\int P_t\varphi\, dl_1 = \int P_t\varphi\, dl_2$. Then, by dominated convergence, for any $\varphi \in \mathcal{C}_b$, $\int \varphi\, dl_1 = \int \varphi\, dl_2$. Therefore L vanishes on $\mathcal{C}_b$ which is dense in $L^1(\gamma_{r,p})$.2
As a consequence, an (r,p)-finite energy measure generating a given (r,p)-potential is uniquely determined by this potential.

Another example of slim set is the following.

Proposition 2.6 *Any countable set is a slim set.*

Proof: It is enough to prove that, if $r > 0$, $p > 1$ and $x \in E$, then $\{x\}$ is (r,p)-polar. By corollary 1.5.2, we have to prove that $\nu(\{x\}) = 0$ for any (r,p)-finite energy measure ν, or equivalently, that the Dirac measure δ_x is not an (r,p)-finite energy measure. Let $(l_n)_{n\geq 0}$ be an orthonormal sequence in $L^2(m)$ consisting of continuous linear functionals on E. We can assume $r \in \mathbb{N}$. By the differential characterization of the (r,p)-norm, we have, if δ_x is an (r,p)-finite energy measure,

$$\exists C \geq 0,\ \ \forall N \in \mathbb{N},\ \ \forall \varphi \in \mathcal{D}(\mathbb{R}^N),$$

$$|\varphi(l_1(x),\cdots,l_N(x))|^p \leq C\left[\int_{\mathbb{R}^N} |\varphi^{(r)}(y)|^p \mathrm{e}^{-|y|^2/2} dy + \int_{\mathbb{R}^N} |\varphi(y)|^p \mathrm{e}^{-|y|^2/2} dy\right].$$

Let $\psi \in \mathcal{D}(\mathbb{R}^N)$ with $\psi(0) = 1$. We apply the above inequality to $\varphi(y) = \psi(n(y_1 - l_1(x), \cdots, y_N - l_N(x)))$. Letting n tend to infinity, we get, if $rp < N$, $1 \leq 0$. 2

We finish this section with some examples of finite energy measures and remarks. Let σ be a measure on $\mathbb{R}^r$ such that there exists $0 \leq \alpha < 1/2$ with $\int \mathrm{e}^{-\alpha|x|^2} \, d\sigma(x) < \infty$. Let $(l_1, \cdots, l_r)$ be a set of r continuous linear functionals on E which is an orthonormal system in $L^2(m)$. We can define by approximation $\sigma(l_1, \cdots, l_r)$ as an (r,p)-finite energy measure for any $p > \frac{1}{1-2\alpha}$ (cf. [2]). In particular, if $l \in E'$ and $\|l\|_2 = 1$, $\delta(l)$ is a $(1,2)$-finite energy measure ($\sqrt{2\pi}\delta(l)$ is the conditionning measure given $\{l = 0\}$). Clearly $\int l^2 \, d(\delta(l)) = 0$, therefore the support of $\delta(l)$ is contained in Ker$l$ (which implies that $\delta(l)$ is singular with respect to m). Thus, Ker$l$ is a closed subspace of E which satisfies $m(\text{Ker}l) = 0$ and $c_{1,2}(\text{Ker}l) > 0$ (because $\delta(l)(\text{Ker}l) = (2\pi)^{-1/2} > 0$). In another direction, it is proved in [8] that if a Borel linear subspace G satisfies $m(G) = 0$ and $\mathcal{H} \subset G$, then G is $(1,p)$-polar for any $p > 1$. Finally, we notice that according to [34], if X is an $\mathbb{R}^d$-valued non degenerate (in Malliavin's sense) Wiener functional and if σ is a temperated measure on $\mathbb{R}^d$, $\sigma(X)$ may be defined and it is a finite energy measure.

3 Multiparameter processes

We introduce,in this section (essentially based on [18]), a class of symmetric Markov multiparameter processes and we show that they allow us to interpret probabilistically some capacities. We shall see that, in particular, the capacities $c_{r,2}$ ($r > 0$) defined in the previous section can be interpreted in such a way.

n-parameter symmetric Markov processes

We fix a metric space E and a Borel probability measure m on E. We also fix a positive integer n. We consider X, an n-parameter E-valued measurable process defined on a probability space $(\Omega, \mathcal{A}, P)$. We begin with some notation.

Notation 3.1 If $B \subset \{1, 2, \ldots, n\}$, we denote by B^c the complement of B in $\{1, 2, \ldots, n\}$. If $t \in \mathbb{R}_+^n$, we set $t_B = (t_i;\ i \in B)$ and we identify $\mathbb{R}_+^n$ with $\mathbb{R}_+^B \times \mathbb{R}_+^{B^c}$ by identifying t with (t_B, t_{B^c}). The order on $\mathbb{R}_+^B$ is the product order and $|t_B|$ denotes $\sum_{i \in B} t_i$.

Definitions 3.1 Process X is called an *E-valued m-symmetric n-parameter Markov process* if there exist $(\mathbb{P}^i;\ 1 \leq i \leq n)$, a family of n strongly continuous semi-groups of sub-Markovian symmetric operators on $L^2(m)$, and $(\mathcal{F}^i;\ 1 \leq i \leq n)$, a family of n filtrations on $(\Omega, \mathcal{A})$ such that

1. $\forall t \in \mathbb{R}_+^n$, $X_t \in \bigcap_{1\leq i\leq n} \mathcal{F}_{t_i}^i$ and the law of X_t is m.
2. $\forall 1 \leq i \leq n$, $\forall f \in L^2(m)$, $\forall u \in \mathbb{R}_+^{\{i\}^c}$, $\forall a, b \in \mathbb{R}_+^{\{i\}}$,

$$E[f(X_{a+b,u}) \mid \mathcal{F}_a^i] = \mathbb{P}_b^i f(X_{a,u}).$$

The semi-groups $(\mathbb{P}^i)$ are called the *transition semi-groups* of X.

Remark The problem to know for which family of semi-groups there exists a process admitting them as transition semi-groups is open. A necessary (but not sufficient, according to [30]) condition is that the semi-groups commute: $\forall i, j \in \{1, \cdots, n\}$, $\forall t_i, t_j \in \mathbb{R}_+$, $\mathbb{P}_{t_i}^i \mathbb{P}_{t_j}^j = \mathbb{P}_{t_j}^j \mathbb{P}_{t_i}^i$. Namely,

$$\mathbb{P}_{t_i}^i \mathbb{P}_{t_j}^j f(X_0) = E[\mathbb{P}_{t_j}^j f(X_{0,t_i}) \mid \mathcal{F}_0] = E[f(X_{0,t_i,t_j}) \mid \mathcal{F}_0]$$

where $\mathcal{F}_0$ denotes $\cap_i \mathcal{F}_0^i$.

Notation 3.2 If $B \subset \{1, 2, \ldots, n\}$, we adopt the following notation: $\mathbb{U} = \prod_{1\leq i\leq n} \frac{1}{\sqrt{\pi}} \int_0^\infty a^{-1/2} \mathrm{e}^{-a} \mathbb{P}_a^i \, da$. If $B \neq \emptyset$, $\mathbb{V}^B = \prod_{i\in B} \int_0^\infty \mathrm{e}^{-a} \mathbb{P}_a^i \, da$ and $\mathbb{V}^\emptyset = I$ (identity in $L^2(m)$). When $B = \{1, \cdots, n\}$, we simply denote $\mathbb{V}^B$ by $\mathbb{V}$ ($\mathbb{V} = \mathbb{U}^2$). For $t \in \mathbb{R}_+^n$, $\mathcal{F}_t = \cap_{1\leq i\leq n} \mathcal{F}_{t_i}^i$. Operator $\mathbb{U}$ is called the $1/2$- *potential operator* of X. Actually, $\mathbb{U} = \prod_{1\leq i\leq n} (I - A_i)^{-1/2}$ where A_i is the infinitesimal generator of $\mathbb{P}^i$.

If $g \in L^2(m)$, $s \in \mathbb{R}_+^n$ and $B \in \{1, \cdots, n\}$, we set

$$M_s^g = E[\int_{t\geq 0} \mathrm{e}^{-|t|} g(X_t) \, dt \mid \mathcal{F}_s], \quad M_\infty^g = \int_{t\geq 0} \mathrm{e}^{-|t|} g(X_t) \, dt,$$

$$\text{if } B \neq \emptyset, \quad H_{B,s}^g = \int_{0\leq t_B\leq s_B} \mathrm{e}^{-|t_B|} \mathbb{V}^{B^c} g(X_{t_B, s_{B^c}}) \, dt_B \text{ and,}$$

if $B = \emptyset$, $H_{B,s}^g = \mathbb{V} g(X_s)$.

In what follows, N_2 denotes the L^2-norm, with respect to P as well as with respect to m.

We have the following easy consequences of the definitions.

Theorem 3.1 *Assume that X is an E-valued m-symmetric n-parameter Markov process. Then*

1. *(Doob's inequality)* $\forall g \in L^2(m)$, *$\forall D$ finite subset of* $\mathbb{R}_+^n$,

$$N_2[\sup_{s\in D} |M_s^g|] \leq 2^n N_2[M_\infty^g]$$

2. *(Generalized Dynkin's formula)* $\forall g \in L^2(m)$, $\forall s \in \mathbb{R}_+^n$,

$$M_s^g = \sum_{B\subset\{1,2,\cdots,n\}} \mathrm{e}^{-|s_{B^c}|} H_{B,s}^g$$

3. $\forall g \in L^2(m)$, *$\forall \varphi$ non negative bounded Borel function on* $\mathbb{R}_+^n$,

$$N_2[M_\infty^g] = N_2[\mathbb{U}g] \text{ and } N_2[\int_{t\geq 0} \mathrm{e}^{-|t|}\varphi(t)g(X_t)\, dt] \leq \|\varphi\|_\infty N_2[\mathbb{U}g].$$

First inequality

In the remainder of this section, we assume that X is an E-valued m-symmetric n-parameter Markov process and that the associated 1/2-potential operator $\mathbb{U}$ is the natural extension to $L^2(m)$ of a Borel kernel $U(x,dy)$ satisfying $U(\mathcal{C}_b) \subset \mathcal{C}_b$. Then, fixing $p = 2$ and identifying $\mathbb{U}$ and $\mathbb{U}_2$ (cf. notation 1.1), hypotheses (H_1), (H_2), (H_3) of section 1 are satisfied. We adopt henceforth the notation of section 1. In particular, we denote by γ and c the capacities associated with U and $p = 2$, and H denotes the space $\mathbb{U}(L^2(m))$. By the symmetry of $\mathbb{U}$ and corollary 1.7.1, we obtain directly:

Proposition 3.1 *For any $u \in \mathcal{P}$, there exists a sequence (p_k) in $L^2(m)^+$ such that* $\lim_k \mathbb{V}p_k = u$ *in H.*

In what follows, if F is any non negative function on Ω, we denote by $E(F)$ the upper integral with respect to P, which means:

$$E(F) = \inf\{E(G);\ \ G \text{ measurable and } G \geq F\}.$$

Finally, we also assume the following weak regularity property:
Right continuity hypothesis: P-a.s., $\forall t\ \ \lim_{t'\downarrow t} X_{t'} = X_t$.
The first inequality is then stated in the following theorem.

Theorem 3.2 *For any function f on E,*

$$E\left[\left[\sup_{t\geq 0} \mathrm{e}^{-|t|}|f|(X_t)\right]^2\right] \leq 4^n[\gamma(f)]^2.$$

Proof: Suppose first $f = \mathbb{V}g$ with $g \in L^2(m)^+$. By theorem 3.1, $\mathrm{e}^{-|s|}\mathbb{V}g(X_s) \leq M_s^g$ and therefore $N_2\left[\sup_{s\in D} \mathrm{e}^{-|s|}\mathbb{V}g(X_s)\right] \leq 2^n N_2(\mathbb{U}g) \leq 2^n\|\mathbb{V}g\|_H$, or, $N_2\left[\sup_{s\in D} \mathrm{e}^{-|s|}f(X_s)\right] \leq 2^n\|f\|_H$.
Suppose then $f \in \mathcal{P}$. By proposition 3.1, the same inequality holds.
Now, if $f \in \mathcal{F}^1(\gamma)$, then $|f| \leq \phi_f$ a.s. and $\|\phi_f\|_H = \gamma(f)$, therefore $N_2\left[\sup_{s\in D} \mathrm{e}^{-|s|}|f|(X_s)\right] \leq 2^n\gamma(f)$.
By right continuity hypothesis, if $f \in \mathcal{F}^1(\gamma)\cap\mathcal{C}$, $N_2\left[\sup_{t\geq 0} \mathrm{e}^{-|t|}|f|(X_t)\right] \leq 2^n\gamma(f)$. This extends to any non negative l.s.c. function by increasing limit.
Finally the general result follows by the definition of γ. 2

Finite energy measures and additive functionals

Besides the previous hypotheses, we henceforth assume that hypothesis (H_4) (tightness of c) and hypothesis (H_5) (density of H in $L^1(\gamma)$) are satisfied.

Theorem 3.3 *Let $\nu \in \mathcal{E}$ be a finite energy measure and denote by u_ν the potential generated by ν. Then there exists a unique random measure $A_\nu(dt)$ on $\mathbb{R}_+^n$ such that, for any sequence (p_k) in $L^2(m)^+$ such that $\lim_k \mathbb{V}p_k = u_\nu$ in H,one has*

$$\forall\varphi \in \mathcal{C}_c(\mathbb{R}_+^n) \quad A_\nu(\varphi) = \lim_k \int \varphi(t)p_k(X_t)\, dt \text{ in } L^2(P).$$

Then, for any non negative Borel function φ (resp. g) on $\mathbb{R}_+^n$ (resp. E),

$$E\left[\int \varphi(t)\, g(X_t)\, A_\nu(dt)\right] = \int \varphi(t)\, dt \int g\, d\nu.$$

Sketch of the proof: By theorem 3.1, for any bounded Borel function φ on $\mathbb{R}_+^n$, for any $g \in L^2(m)$,

$$N_2[\int \mathrm{e}^{-|t|}\varphi(t)g(X_t)\, dt] \leq 2\|\varphi\|_\infty\|\mathbb{V}g\|_H.$$

Then A_ν can be defined as a vague limit (weak limit) of some subsequence $p_{k'}(X_t)\, dt$ (in this sense, $T \longrightarrow A_\nu([0,T])$ is an "additive functional").
For the second part of the statement, we remark that if $p \in L^2(m)^+$,

$$E[\int \varphi(t)g(X_t)p(X_t)\, dt] = \int \varphi(t)\, dt \int g\, p\, dm.$$

We have to pass to the limit in this formula, but there are some technical difficulties, and, in particular, we need the right continuity hypothesis and additional assumption (H_5). For details, we refer to [18]. 2

Second inequality

Under the same assumptions as in the previous theorem, we have:

Theorem 3.4 *For any Borel function f on E, for any $T \in \mathbb{R}_+^n$,*

$$\left(\int_{[0,T]} \mathrm{e}^{-|t|}dt\right)^2 [\gamma(f)]^2 \leq E\left[\left[\sup_{t\in[0,T]} |f|(X_t)\right]^2\right].$$

Proof: By proposition 1.9, it suffices to consider the case where f is a non negative u.s.c. function with compact support. Let then ν be the equilibrium measure of f and let A_ν be the associated random measure (theorem 3.3). By theorem 1.5, we obtain

$$\left(\int_{[0,T]} \mathrm{e}^{-|t|}dt\right)\gamma^2(f) = \int_{[0,T]} \mathrm{e}^{-|t|}dt \int f\, d\nu = E[\int_{[0,T]} \mathrm{e}^{-|t|} f(X_t)\, A_\nu(dt)]$$

$$\leq N_2\left(\sup_{t\in[0,T]} |f(X_t)|\right)\, N_2\left(\int_{[0,T]} \mathrm{e}^{-|t|} A_\nu(dt)\right).$$

Now, by the definition of A_ν,

$$N_2\left(\int_{[0,T]} \mathrm{e}^{-|t|} A_\nu(dt)\right) \leq \|u_\nu\|_H = \gamma(f).$$

The result follows. 2

As a consequence of theorems 3.2 and 3.4, we have:

Corollary 3.4.1 *For any Borel subset B of E, for any $T \in \mathbb{R}_+^n$,*

$$\left(\int_{[0,T]} \mathrm{e}^{-|t|}dt\right)^2 [c(B)]^2 \leq P[\exists t \in [0,T];\;\; X_t \in B] \leq 4^n \mathrm{e}^{2|T|}[c(B)]^2.$$

In particular, a Borel set B is polar if and only if, almost surely, $X_t \notin B$ for any t.

These results can be used to give a probabilistic characterization of quasi-continuity (see [18]).

Capacities $c_{n,2}$, $n \in \mathbb{N}^*$

We now consider the framework of section 2, and we fix $n \in \mathbb{N}^*$. In particular, E denotes the Wiener space and m is the Wiener measure.

Let $W^{(n+1)}$ be an $\mathbb{R}^d$-valued $(n+1)$-parameter Brownian sheet defined on a probability space $(\Omega, \mathcal{A}, P)$. We set

$$X_t^{(n)} = \mathrm{e}^{-|t|/2} W^{(n+1)}_{\mathrm{e}^{t_1}, \cdots, \mathrm{e}^{t_n}, \cdot}$$

Then $X^{(n)}$ is an E-valued n-parameter process, called the *E-valued n-parameter Ornstein-Uhlenbeck process* (see [25] for $n = 1$, [33] for $n = 2$, and [29], [14] for the general case). The following proposition follows easily from the definitions.

Proposition 3.2 *Process $X^{(n)}$ is an E-valued m-symmetric n-parameter Markov process with continuous paths. Its transition semi-groups are given by:*

$\forall 1 \leq i \leq n \ \ \forall t > 0 \quad \mathbb{P}_t^i = \mathbb{P}_{t,2}$ *(Ornstein-Uhlenbeck semi-group on $L^2(m)$).*

The 1/2-potential operator of $X^{(n)}$ then is operator $\mathbb{U}_2^n$ and the corresponding capacity is $c_{n,2}$. Consequently, all assumptions of the previous paragraph are satisfied. In particular, corollary 3.4.1 is valid with $E =$ the Wiener space, $c = c_{n,2}$ and $X = X^{(n)}$. This situation was generalized in [1],[9].

Capacities $c_{n+\alpha,2}$, $n \in \mathbb{N}$, $0 < \alpha < 1$

The framework and the notation are the same as in the previous paragraph. We set $r = n + \alpha$. Let $X^{(n+1)}$ be the E-valued $(n+1)$-parameter Ornstein-Uhlenbeck process. We consider $(\tau_t)_{t\geq 0}$ a one-sided stable process of index α, starting from 0, with càd-làg paths and independent of $W^{(n+2)}$. We set

$$Y^{(r)}_{t_1,\cdots,t_{n+1}} = X^{(n+1)}_{t_1,\cdots,t_n,\tau_{t_{n+1}}}$$

Process $Y^{(r)}$ is an E-valued $(n+1)$-parameter right continuous process. We set, as before,

$$\forall 1 \leq i \leq n \quad \mathbb{P}_t^i = \mathbb{P}_{t,2}$$

and $\mathbb{P}_t^{n+1} = \displaystyle\int \mathbb{P}_{s,2}\, d\nu_t(s)$ where ν_t is the law of τ_t.

Proposition 3.3 *Process $Y^{(r)}$ is an E-valued m-symmetric $n+1$-parameter Markov process of which the transition semi-groups are* $(\mathbb{P}^i;\ 1 \leq i \leq n+1)$.

The proof is almost classical (cf. [17], [18]). The 1/2-potential operator of $Y^{(r)}$ is $\mathbb{U}_2^n(I + (-A_2)^\alpha)^{-1/2}$, where A_2 is the Ornstein-Uhlenbeck operator in $L^2(m)$, and then the corresponding space H is $\mathbb{D}_2^r$ with an equivalent norm. The associated capacity is therefore equivalent to $c_{r,2}$. The assumptions (H_1) to (H_5) are again satisfied. In particular, for any $T > 0$, there exist a_T and b_T, positive constants, such that, for any Borel subset B of E,

$$a_T[c_{r,2}(B)]^2 \leq P[\exists t \in [0,T]^{n+1};\ \ Y_t^{(r)} \in B] \leq b_T[c_{r,2}(B)]^2.$$

Remark In the context of the previous examples, we can prove more precise properties of the random measure A_ν associated with a finite energy measure ν, in particular $T \in \mathbb{R}_+^n \longrightarrow A_\nu([0,T])$ is almost surely continuous (see [19]).

4 Quasi-sure analysis

The quasi-sure analysis (terminology introduced by P. Malliavin) is the study of properties up to a slim set (in place of negligible set). The interest comes from the fact that the finite energy measures, which appear in particular in the context of conditionning, do not charge slim sets but may be singular with respect to the Wiener measure (see at the end of section 2). A useful tool is to consider spaces of Banach-valued functions (we refer to [7], [2], [4, 5], [24], [28]). In fact, there are different definitions which are not equivalent in general.

Banach-valued functions

This paragraph is close to the work of L. Denis [4, 5], but the context is slightly different and we also adopt slightly different definitions.

The context here is that of the first section: We consider (E, m, U) satisfying (H_1) and (H_2). We fix $1 < p < \infty$ and we assume (H_3) too. The notation is that of section 1. We also fix a separable Banach space $\mathbb{B}$. It is clear that U can be naturally extended to $L^p(m; \mathbb{B})$.

Notation 4.1 We denote by $\overline{\mathbb{U}}$ the natural extension of U to $L^p(m; \mathbb{B})$ (operator $\overline{\mathbb{U}}$ is a contraction in $L^p(m; \mathbb{B})$).

Proposition 4.1 *Operator $\overline{\mathbb{U}}$ is injective on $L^p(m; \mathbb{B})$.*

This is an easy consequence of the Hahn-Banach theorem and of the property:

$$\forall F \in L^p(m; \mathbb{B})\ \forall \varphi \in \mathbb{B}' \quad \varphi(\overline{\mathbb{U}}(F)) = \overline{\mathbb{U}}(\varphi(F)).$$

Notation 4.2 We denote by $H^{\mathbb{B}}$ the image $\overline{\mathbb{U}}(L^p(m;\mathbb{B}))$ equipped with the norm

$$\|\overline{\mathbb{U}}F\|_{H^{\mathbb{B}}} = \|F\|_{L^p(m;\mathbb{B})}.$$

Therefore, $H^{\mathbb{B}}$ is a separable Banach space isometric to $\mathcal{L}^p(m;\mathbb{B})$.

We then define $\mathcal{F}^1(\gamma;\mathbb{B})$ as the set of functions $f: E \longrightarrow \mathbb{B}$, defined q.e., such that $\|f\|_{\mathbb{B}} \in \mathcal{F}^1(\gamma)$. For such a function f, we set $\gamma(f) = \gamma(\|f\|_{\mathbb{B}})$.

We set

$$\mathcal{L}^1(\gamma;\mathbb{B}) = \{f \in \mathcal{F}^1(\gamma;\mathbb{B});\ \exists(\varphi_k) \in \mathcal{C}_b(E;\mathbb{B})\ \ \gamma(f-\varphi_k) \to 0\},$$

and we define $L^1(\gamma;\mathbb{B})$ as the quotient space with respect to the q.e. equality.

A function $f: E \longrightarrow \mathbb{B}$ is said to be quasi-continuous if there is a nest (F_k) such that, for any k, $f|_{F_k} \in \mathcal{C}(F_k;\mathbb{B})$. We obtain easily, as in the scalar case, the following proposition.

Proposition 4.2 *1. Each element of $\mathcal{L}^1(\gamma;\mathbb{B})$ is quasi-continuous.*

2. $L^1(\gamma;\mathbb{B})$ is a Banach space.

One of the main results is the following extension of the scalar case (theorem 1.1).

Theorem 4.1 *Any $h \in H^{\mathbb{B}}$ admits a quasi-continuous m-representative $\tilde{h}$, unique up to quasi-everywhere equality, and*

$$\tilde{h} \in \mathcal{L}^1(\gamma;\mathbb{B}) \text{ and } \gamma(\tilde{h}) \leq \|h\|_{H^{\mathbb{B}}}.$$

Proof: As $\mathcal{C}_b(E;\mathbb{B})$ is dense in $L^p(m;\mathbb{B})$, then $\mathcal{C}_b(E;\mathbb{B}) \cap H^{\mathbb{B}}$ is dense in $H^{\mathbb{B}}$. Now, for any $f = \overline{\mathbb{U}}g$ in $\mathcal{C}_b(E;\mathbb{B}) \cap H^{\mathbb{B}}$, by theorem 1.1,

$$\gamma(f) \leq \gamma(\widetilde{\mathbb{U}\|g\|_{\mathbb{B}}}) \leq \|f\|_{H^{\mathbb{B}}}.$$

The result follows. 2

Hence, as in the scalar case, $H^{\mathbb{B}}$ may be considered as a subspace of $L^1(\gamma;\mathbb{B})$. We also have:

Theorem 4.2 *Assume that (H_4) is satisfied. Then $f \in \mathcal{L}^1(\gamma;\mathbb{B})$ if and only if $f \in \mathcal{F}^1(\gamma;\mathbb{B})$ and f is quasi-continuous. If in addition (H_5) is satisfied, then $H^{\mathbb{B}}$ is dense in $L^1(\gamma;\mathbb{B})$.*

Proof: Consider first a bounded $\mathbb{B}$-valued function f which is quasi-continuous. By (H_4), there exists a nest (K_k) consisting of compact sets, such that, for any k, $f|_{K_k}$ is continuous. It is easy to see that the algebraic tensor product $\mathcal{C}(K_k) \otimes \mathbb{B}$ is dense in $\mathcal{C}(K_k; \mathbb{B})$. Therefore, there exists $\varphi_k \in \mathcal{C}(K_k) \otimes \mathbb{B}$ such that $\|f - \varphi_k\|_{\mathbb{B}} \leq \varepsilon \leq 1$ on K_k. By extension, we may assume that $\varphi_k \in \mathcal{C}_b(E) \otimes \mathbb{B}$ and $\|\varphi_k\|_\infty \leq \|f\|_\infty + 1$. If (H_5) is satisfied, there exists $h_k \in H^{\mathbb{B}}$ such that $\gamma(\varphi_k - h_k) \leq \varepsilon$. We then have

$$\gamma(f - \varphi_k) \leq \varepsilon + (2\|f\|_\infty + 1)c(K_k^c) \text{ and } \gamma(f - h_k) \leq \gamma(f - \varphi_k) + \varepsilon.$$

This proves that $f \in \mathcal{L}^1(\gamma; \mathbb{B})$ and, if (H_5) is satisfied, any bounded function in $\mathcal{L}^1(\gamma; \mathbb{B})$ can be approximated by elements of $H^{\mathbb{B}}$.
The general case may be obtained by the following remark which can be proved as in proposition 1.5: If $f \in \mathcal{F}^1(\gamma; \mathbb{B})$ and $f_n = n(n \vee \|f\|_{\mathbb{B}})^{-1} f$, then $\gamma(f - f_n)$ tends to 0 when n tends to infinity. Now, f_n is bounded, and quasi-continuous if so is f. 2
Theorem 4.1 is a useful tool to transform almost-sure convergence results into quasi-sure convergence results. We follow L. Denis ([4, 5]).

Theorem 4.3 *Let $(f_n)_{n\geq 0}$ be a sequence of H. Let, for any n, $g = \mathbb{U}^{-1} f_n$. We assume*

1. *$\exists g_\infty$ such that $\lim_{n\to\infty} g_n = g_\infty$ a.s.*
2. *$\sup_n |g_n| \in L^p(m)$.*

Then, setting $f_\infty = \mathbb{U} g_\infty$, we have

$$\lim_{n\to\infty} \widetilde{f_n} = \widetilde{f_\infty} \text{ q.e.}$$

Proof: Let $\overline{\mathbb{N}}$ be the compact set $\mathbb{N} \cup \{\infty\}$, and $\mathbb{B} = \mathcal{C}(\overline{\mathbb{N}})$. We may consider $g.$ as an element of $L^p(m; \mathbb{B})$ and $f. = \overline{\mathbb{U}}(g.)$. Therefore $f. \in H^{\mathbb{B}}$ and hence $f.$ admits a quasi-continuous representative $\widetilde{f.}$. Clearly, $(\widetilde{f.})_n = \widetilde{f_n}$ q.e. for any $n \in \overline{\mathbb{N}}$. Therefore $\widetilde{f.}(\omega) \in \mathcal{C}(\overline{\mathbb{N}})$ for quasi-every ω and the result follows. 2
In the same way, the continuous analogue holds too:

Theorem 4.4 *Let $T \in [0, \infty]$ and let $(f_t)_{t\in[0,T]}$ be a family in H. We assume that there exists a family of functions $(g_t)_{t\in[0,T]}$ such that*

1. *$\forall t \in [0, T]$, g_t is an m-representative of $\mathbb{U}^{-1} f_t$,*
2. *For almost all $\omega \in E$, $t \longrightarrow g_t(\omega) \in \mathcal{C}([0, T]$,*
3. *$\sup_{t\in[0,T]} |g_t| \in \mathcal{L}^p(m)$.*

Then, there is a family $(\widetilde{f}_t)_{t\in[0,T]}$ *such that*

- *for any* $t \in [0,T]$, $\widetilde{f}_t$ *is a quasi-continuous m-representative of* f_t,
- *for quasi-every* ω, $t \longrightarrow \widetilde{f}_t(\omega) \in \mathcal{C}([0,T])$.

More precisely, $\widetilde{f}_. \in \mathcal{L}^1(\gamma;\mathcal{C}([0,T]))$ *and*

$$\gamma(\sup_t |\widetilde{f}_t|) \leq \| \sup_t |g_t| \, \|_p.$$

Following the same ideas, we can prove a quasi-sure version of Kolmogorov's theorem.

Theorem 4.5 *Let* $(X_t)_{t\in[0,1]^d}$ *be a family in* H. *We assume*

$$\exists C > 0,\ \exists \varepsilon > 0,\ \forall s,t \in [0,1]^d \quad \|X_t - X_s\|_H^p \leq C|t-s|^{d+\varepsilon}.$$

Then there exists $(\widetilde{X_t})_{t\in[0,1]^d}$ *such that*

- $\forall t \in [0,1]^d$ $\widetilde{X_t} = X_t$ *m-a.s.*
- $\forall t \in [0,1]^d$ $\widetilde{X_t}$ *is quasi-continuous*
- *for quasi-every* $\omega \in E$, $t \longrightarrow \widetilde{X_t}(\omega)$ *is Hölder continuous of order* α *for any* $\alpha \in [0, \varepsilon/p[$.

Proof: Let for $\alpha \in [0,\varepsilon/p[$, $\mathcal{H}^\alpha$ be the space of continuous functions f on $[0,1]^d$, nul at 0, such that $\lim_{|s-t|\to 0, s\neq t} |t-s|^{-\alpha}|f(t)-f(s)| = 0$, equipped with the usual norm

$$\|f\|_{\mathcal{H}^\alpha} = \sup_{s\neq t} \frac{|f(t)-f(s)|}{|t-s|^\alpha}.$$

We can assume $X_0 = 0$. Define $Y_t = \mathbb{U}^{-1}X_t$. Then $Y_t \in L^p$ and $\|Y_t - Y_s\|_p^p \leq C|t-s|^{d+\varepsilon}$. Therefore, by the classical Kolmogorov theorem, there exists a version of Y, still denoted by Y, which belongs to $L^p(m;\mathcal{H}^\alpha)$. Clearly $\hat{X} = \overline{\mathbb{U}}(Y)$ is a version of X belonging to $H^{\mathcal{H}^\alpha} \subset \mathcal{L}^1(\gamma;\mathcal{H}^\alpha)$. 2

Still following L. Denis ([4, 5]), we now give applications to Wiener space.

Applications to Wiener space

We use here the results of the previous paragraph in the framework of section 2. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration of $(B_t)_{t\geq 0}$.

Brownian martingales

Theorem 4.6 *Let $f \in \mathcal{L}^1(\gamma_{r,p})$. Then, there exists $F \in \mathcal{L}^1(\gamma_{r,p}; \mathcal{C}([0,\infty]))$ such that, for any $t \geq 0$, F_t is an m-representative of $E(f \mid \mathcal{F}_t)$ and $F_\infty = f$ (r,p)-quasi-everywhere. Consequently, for any $t \in [0,\infty]$, F_t is (r,p)-quasi-continuous, and, for (r,p)-quasi-every ω, $t \in [0,\infty] \longrightarrow F_t(\omega)$ is continuous. Moreover, following capacitary Doob's inequality holds:*

$$\gamma_{r,p}(\sup_t |F_t|) \leq \frac{p}{p-1}\gamma_{r,p}(f).$$

(See [12] for a similar, slightly weaker, result by another method.)
Proof: First assume $f \in \mathbb{D}_p^r$. There exists $g \in L^p$ such that $f = \mathbb{U}_p^r g$. It is easy to see that $\mathbb{U}_p^r$ commutes with $E(\ \mid \mathcal{F}_t)$. Then, $E(f \mid \mathcal{F}_t) = \mathbb{U}_p^r(E(g \mid \mathcal{F}_t))$. By Doob's inequality, there exists $G \in L^p(m; \mathcal{C}([0,+\infty]))$ such that, for any $t \geq 0$ $G_t = E(g \mid \mathcal{F}_t)$ a.s. and

$$\|G\|_{L^p(m;\mathcal{C}([0,+\infty]))} \leq \frac{p}{p-1}\|g\|_p.$$

Therefore theorem 4.4 applies and the first part of the statement of the theorem holds. Let φ be the (r,p)-equilibrium potential of f, $\varphi = \mathbb{U}_p^r\psi$. We can associate as before Φ with φ. Clearly, (r,p)-quasi-everywhere, for any t $|F_t| \leq \Phi_t$. Therefore, by theorem 4.4

$$\gamma_{r,p}(\sup_t |F_t|) \leq \gamma_{r,p}(\sup_t \Phi_t) \leq \frac{p}{p-1}\|\psi\|_p = \frac{p}{p-1}\|\varphi\|_{r,p}$$

and $\|\varphi\|_{r,p} = \gamma_{r,p}(f)$.
The general result then follows from the density of $\mathbb{D}_p^r$ in $L^1(\gamma_{r,p})$. 2

Quadratic variation

We fix $T > 0$. For any subdivision $\Delta = \{0 = t_0 \leq t_1 \cdots \leq t_n = T\}$ of the interval $[0,T]$, we denote by S_Δ the $d \times d$-matrix

$$S_\Delta = \sum_{i=0}^{n-1}(B_{t_{i+1}} - B_{t_i})(B_{t_{i+1}} - B_{t_i})^*$$

(where $(B_{t_{i+1}} - B_{t_i})^*$ denotes the transposed matrix of the column matrix $(B_{t_{i+1}} - B_{t_i})$. We denote by I_d the $d \times d$-identity matrix.

Theorem 4.7 *Let $(\Delta_n)_{n\geq 0}$ be a sequence of subdivisions of $[0,T]$ such that*

$$\lim_{n\to\infty} S_{\Delta_n} = TI_d \ m\text{-}a.s.$$

Then, $\lim_{n\to\infty} S_{\Delta_n} = TI_d$ outside a slim set.

(The same result was proved by Feyel-de La Pradelle ([7]), using their result on sublinear functionals (proposition 2.4).)
Proof: It is easy to see that, for $r > 0$,

$$S_\Delta = U^r(2^{r/2}(S_\Delta - TI_d) + TI_d).$$

On the other hand, by Fernique's theorem, if $\lim_n S_{\Delta_n} = TI_d$ m-a.s., $\sup_n |S_{\Delta_n}| \in L^p$ (where $|\quad|$ denotes a norm on $\mathbb{R}^{d\times d}$). We may then apply theorem 4.3. 2

Stochastic differential equations

We fix $T > 0$, $n \in \mathbb{N}^*$, and we denote by $\mathbb{B}$ the Banach space $\mathcal{C}([0,T];\mathbb{R}^n)$. We begin with a preliminary result (cf. [7]).

Proposition 4.3 *Let* $\alpha_\cdot \in L^p([0,T];(\mathbb{D}_p^r)^{n\times d})$ *be an* $\mathbb{R}^{n\times d}$*-valued adapted process. Then* $\int_0^\cdot \alpha_s\, dB_s$ *belongs to* $(\mathbb{D}_p^r)^{\mathbb{B}}$.

Sketch of the proof: Set

$$\widehat{\mathbb{U}}^r = \frac{1}{\Gamma(r/2)} \int \mathrm{e}^{-(3/2)t} t^{(r/2)-1} \mathbb{P}_t \, dt.$$

There exists an adapted process $\beta \in L^p([0,T]);(L^p)^{n\times d})$ such that $\alpha_s = \widehat{\mathbb{U}}^r(\beta_s)$. Then $\int_0^\cdot \beta_s\, dB_s \in L^p([0,T];\mathbb{B})$ and

$$\int_0^\cdot \alpha_s\, dB_s = \overline{\mathbb{U}^r}(\int_0^\cdot \beta_s\, dB_s).$$

2

We then obtain by successive approximations the following result which was obtained by other methods in [32], [27], ...

Theorem 4.8 *Let* σ *(resp.* b*) be a* $\mathcal{C}_b^\infty$ *function from* $\mathbb{R}^n$ *into* $\mathbb{R}^{n\times d}$ *(resp.* $\mathbb{R}^n$*). For* $x \in \mathbb{R}^n$*, we consider the continuous strong solution* X *of*

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad X_0 = x.$$

Then, for any $r > 0$*,* $p > 1$*,* $X \in (\mathbb{D}_p^r)^{\mathbb{B}}$*. As a consequence, there exists a version of* X *belonging to* $\cap_{r>0,p>1}\mathcal{L}^1(\gamma_{r,p};\mathbb{B})$.

Quasi-sure continuity of the Ornstein-Uhlenbeck semi-group

Theorem 4.9 *Let* $f \in \mathcal{L}^1(\gamma_{r,p})$*. Then* $\lim_{t\to 0} P_t f = f$ (r,p)*-q.e. More precisely, for any* $T > 0$, $\omega \in E \longrightarrow (t \in [0,T] \to P_t f(\omega))$ *belongs to* $\mathcal{L}^1(\gamma_{r,p}; \mathcal{C}([0,T]))$.

Proof: Suppose first $f \in \mathbb{D}^r_p$. Then $f = \mathbb{U}^r_p g$ with $g \in L^p$ and $P_t f = U^r P_t g$ (r,p)-q.e. By [31], hypotheses of theorem 4.4 are satisfied with $f_t = P_t f$, $g_t = P_t g$. As, for any t, $f_t \in \mathcal{L}^1(\gamma_{r,p})$, the result is obtained in this case and $\gamma(\sup_t |P_t f|) \leq C\|f\|_{r,p}$. We can then proceed as in the proof of theorem 4.6.

2

Many other results of quasi-sure analysis can be found in [5]. Nevertheless, it is not always true that an m-a.s. convergence theorem admits a quasi-sure version. For example, if $d = 3$ or 4, $\lim_{t\to\infty} |B_t(\omega)| = \infty$ m-a.s., but it is proved in [22] that, for any $\varepsilon > 0$,

$$c_{1,2}(\{\omega;\ \liminf_{t\to\infty} |B_t(\omega)| < \varepsilon\}) > 0.$$

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