# Modelling of Default Risk: An Overview

Monique Jeanblanc* Marek Rutkowski†

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**Abstract**

The aim of these notes is to provide a relatively concise - but still self-contained - overview of mathematical notions and results which underpin the valuation of defaultable claims. Though the default risk modelling was extensively studied in numerous recent papers, it seems nonetheless that some of these papers lack a sound theoretical background. Our goal is to furnish results which cover both the classic *value-of-the-firm* (or *structural*) approach, as well as the more recent *intensity-based* methodology.

The notes are organized as follows. In Section 1, we provide a brief introduction to the default risk modelling, and to the associated mathematical concepts. Section 2 in entirely devoted to an exposition of various structural models, in which the default event is related to a hitting time to a constant or variable barrier. In Section 3, we give a brief overview of various models which are derived within the so-called intensity-based approach.

Subsequently, in Section 4, we provide a detailed analysis of the relatively simple case when the flow of informations available to an agent reduces to the observations of the random time which models the default event. The focus is on the evaluation of conditional expectations with respect to the filtration generated by a default time with the use of the intensity function.

*Equipe d'Analyse et Probabilités, Université d'Evry Val d'Essonne, Boulevard des Coquibus, 91025 Evry Cedex, France; e-mail : jeanbl@maths.univ-evry.fr

†Institute of Mathematics, Warsaw University of Technology, pl.Politechniki 1, 00-661 Warszawa, Poland; e-mail: markrut@mini.pw.edu.pl

Results of Section 4 are then generalized in Section 5 to the case when an additional information flow - formally represented by some filtration $\mathbf{F}$ - is present. At the intuitive level, $\mathbf{F}$ is generated by prices of some assets, or by other economic factors (e.g., interest rates). Though in typical examples $\mathbf{F}$ is chosen to be the Brownian filtration, most theoretical results obtained in Section 5 do not rely on such a specification of the filtration $\mathbf{F}$. Special attention is paid here to the hypothesis (H), which postulates the invariance of the martingale property with respect to the enlargement of $\mathbf{F}$ by the observations of a default time.

In Section 6, we examine several non-trivial examples of the calculation of the stochastic intensity of a default time (or rather of the dual predictable projection of the associated first jump process). Since in this section the underlying filtration $\mathbf{F}$ is assumed to be generated by a Brownian motion, and it is well known that all stopping time with respect to the Brownian filtration are predictable (so that they do not admit intensity with respect to $\mathbf{F}$), it is natural to examine random times which are not $\mathbf{F}$-stopping times. To be more specific, we study last passage times of a Brownian motion (with or without drift), and more general, of a diffusion process.

Finally, in Section 7, we continue the general study of the properties of stochastic intensity, focusing on the case the minimum of several random times. The problem of finding the intensity of the minimum of several random times appears in a natural way, for instance, in the so-called *first-to-default* valuation of defaultable debt.

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# 1 Introduction

In an arbitrage-free complete financial market, the $t$-time price $X_t$ of a promised payoff $X$ paid at a terminal time $T$ is

$$X_t = E\Big(X \exp\Big(-\int_t^T r_u\,du\Big)\,\Big|\,\mathcal{F}_t\Big),$$

where $(r_s, s \geq 0)$ is the spot interest rate. In the formula above, the expectation is computed under the so-called *equivalent martingale measure* (e.m.m.) which, under some technical hypotheses, is unique, and $\mathcal{F}_t$ is the information available at time $t$. The proof of this result relies on a replication tool: with an initial investment equal to $x = X_0$, a financial agent is able to buy a self-financing portfolio with terminal value equal to $X$, i.e., a hedging portfolio, and the $t$-time value of this portfolio is $X_t$. If the market is not complete, there exist contingent claims that cannot be hedged. Let us stress that it is important to explicitly specify the information available to the agent.

In the default risk framework, a default appears at some random time $\tau$. Let us denote by $1\!\!1_{\{T<\tau\}}$ the indicator function of the set $\{T < \tau\}$, equal to 1 if the default occurs after $T$ and equal to 0 otherwise. A default free contingent claim consists of a nonnegative random variable which represent the amount of cash paid at a prespecified time $T$ to the owner of the claim. For a defaultable contingent claim, the promised payment is actually done only if the default did not occur before maturity. If the default occurs before maturity, the payment is not done, and the defaultable claim is worthless after default time. More generally, the payment of a defaultable claim consists of two parts:
1) Given a maturity date $T > 0$, a random variable $X$, which does not depend on $\tau$ represents the promised payoff – that is, the amount of cash the owner of the claim will receive at time $T$, provided that the default has not occured before the maturity date $T$.
2) A predictable process $h$, prespecified in the default-free world, models the payoff which is received if default occurs before maturity. This process is called the recovery process or the rebate.

The value of the defaultable claim is, provided that the default has not occured before time $t$,

$$X_t = E\Big(X 1\!\!1_{\{T<\tau\}} \exp\Big(-\int_t^T r_u\,du\Big) + h_\tau 1\!\!1_{\{\tau\leq T\}} \exp\Big(-\int_t^\tau r_u\,du\Big)\,\Big|\,\mathcal{G}_t\Big),$$

where $\mathcal{G}_t$ is the information at time $t$. We assume that the agent knows when the default appears. At time $t$, the agent knows if the default has occured before; if the default has not yet occured, he has no information on the time when it will happen.

The problem of modelling a default time is well represented in the literature. There are two main approaches: either the default time $\tau$ is a stopping time in the asset's filtration, or it is a stopping time in a larger filtration. The papers of Cooper and Martin [11] and Rogers [49] contain a comparative study of these approaches. The main difference between the two methods is that in the first approach the default time is "announced," whereas it is not the case in the second case.

In the first approach, the so-called *structural approach,* pioneered by Merton [44], the default time $\tau$ is a stopping time in the filtration of the prices. Therefore the valuing of the defaultable claim reduces to the problem of the pricing of the claim $X\mathbb{1}_{\{T<\tau\}}$ which is measurable with respect to the filtration of the prices at time $T$. This is a standard, though difficult, problem, which reduces - in a complete market case - to the computation of the expectation of the discounted payoff under the risk-neutral probability measure.

In the second method, known as the *intensity-based approach,* the aim is also to compute the value of the defaultable claim $X\mathbb{1}_{\{T<\tau\}}$; however it may happen that this claim is not measurable with respect to the $\sigma$-algebra generated by prices up to time $T$. In this case, it is generally assumed that the market is complete for the large filtration, which means that the defaultable claim is hedgeable. In order to compute the expectation of $X\mathbb{1}_{\{T<\tau\}}$ under the risk-neutral probability, it is convenient to introduce the notion of intensity of the default. Then, under some assumptions, the intensity of the default time acts as a change of the spot interest rate in the pricing formula.

We proceed somewhat differently, namely, our goal is to examine the connection between the default-free world and the defaultable one. We recall some well known - though perhaps forgotten - tools to compute this expectation, which simplify most of the proofs in existing literature on default risk modelling. We also try to understand better the meaning of "the information of the agent" and to make precise the relation between the default time and the price's filtration by means of the hazard process.

First, we recall that if the information is only the time when the default appears, the computation of the expectation of a defaultable payoff involves the hazard function of $\tau$. In this case, the compensator of the default process $N_t = \mathbb{1}_{\{\tau \le t\}}$ can be explicitly expressed in terms of the cumulative distribution function of $\tau$. We discuss a result of Duffie and Lando [19] and we give a shorter proof of this result (as well as a simpler form of the intensity of the hitting time).

Subsequently, we assume that the information of the agent at time $t$ consists of knowledge of the behaviour of the prices up to time $t$ as well as the default time. We show that in this case the results depend strongly on the dependence between the asset process and the default time. In particular, we show that the intensity does not provide a sufficient information as far as this link is concerned. We use some tools from the theory of enlargement of filtrations in order to compute the compensator of $N$, provided that it exists. Finally, we give some examples where the usual assumptions made in the literature are not satisfied and where the value of a default claim is not obtained by a change of spot rate.

These notes are partially based on the working papers by Elliott, Jeanblanc and Yor [22] and Rutkowski [50]. Monique Jeanblanc thanks the participants to Aspet's, INRIA's and Ulm workshops for stimulating discussions, as well as the organizers and participants to ISFMA Summer School in Shanghai. Henri Pagès made a careful reading of a first version and corrected a lot of misprints. The remaining errors are ours.

- **Some notation**

We shall write $\mathbf{F}$ to denote a filtration $(\mathcal{F}_t, t \ge 0)$. A process is said to be *càdlàg*[1] (resp. *càglàd*) if it has right continuous paths with left limits (left continuous paths with right limits). If $X$ is a càdlàg process, we denote by $X_{t-}$ the limit of $X_s$ when $s$ goes to $t$, and $s < t$, and by $\Delta X_t = X_t - X_{t-}$ the jump of $X$ at time $t$.

The predictable $\sigma$-algebra $\mathcal{P}$ on $(I\!\!R^+ \times \Omega, \mathcal{B} \times \mathcal{F}_\infty)$ is the smallest $\sigma$-algebra making all adapted càglàd processes measurable. This $\sigma$-algebra is generated by the processes of the form $\mathbb{1}_{]a,b]} F_a$, where $F_a \in \mathcal{F}_a$. In particular, any càglàd adapted process is manifestly predictable.

A semi-martingale is a process $X$ which admits the decomposition $X = M + A$, where $M$ is a martingale and $A$ stands for a predictable process with bounded variation.

[1]This is a French acronym for continu à droite, limites à gauche

The indicator function of the set $B$ is denoted by $\mathbb{1}_B$.

**• Background on stopping times**

We recall few basic definitions related to stopping times. The notion of a stopping time depends on the choice of the filtration. Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be a probability space with a filtration $\mathbf{F} = (\mathcal{F}_t)_{\{t\geq 0\}}$. An $I\!R_+ \cup \{+\infty\}$ valued random variable $\tau$ is a **F**-*stopping time* if $\{\tau \leq t\} \in \mathcal{F}_t$ for any $t$. Obviously, if **G** is a filtration larger than **F**, i.e., $\mathcal{F}_t \subset \mathcal{G}_t$ for any $t$, and $\tau$ is a **F**-stopping time, then $\tau$ is a **G**-stopping time. A stopping time $\tau$ is **F**-*predictable* if there exists an increasing sequence of **F**-stopping times $\tau_n$ such that $\tau_n < \tau$ on $\{\tau > 0\}$ and $\lim \tau_n = \tau$. A stopping time $\tau$ is **F**-*totally inaccessible* if for any **F**-predictable stopping time $S$, $P\{\omega \in \Omega : \tau(\omega) = S(\omega) < \infty\} = 0$.

In a Brownian filtration, it can be proved that any stopping time is a predictable stopping time. The most important example of totally inaccessible stopping time is the first time when a Poisson process jumps.

If $\tau$ is a nonnegative random variable on some probability space $(\Omega, \mathcal{G}, P)$ it is possible to endow $\Omega$ with a filtration such that $\tau$ is a stopping time. This filtration is not unique, and the smallest filtration satisfying this property is $\mathcal{H}_t = \sigma(\{\tau \leq u\} : u \leq t) = \sigma(\sigma(\tau) \cap \{\tau \leq t\})$.

**• Background on stochastic calculus**

Let us recall few basic facts on stochastic processes, which we shall use in what follows. A more detailed account can be found, for instance, in Dellacherie and Meyer [16] or Protter [46].

The *Doob-Meyer decomposition* theorem states that, under suitable integrability assumptions, any supermartingale $Z$ admits a decomposition $Z = M - A$, where $M$ is a local martingale and $A$ an increasing predictable process, with $A_0 = 0$.

A *standard Poisson process* (with constant intensity $\lambda$) is defined as $\widetilde{N}_t = \sum_n \mathbb{1}_{\{T_n \leq t\}}$, where $T_i$ are random variables such that $T_0 = 0$ and $(T_i - T_{i-1}, i \geq 1)$ are i.i.d. random variables with exponential law of parameter $\lambda t$. In this case, $\widetilde{N}_t$ is a random variable with Poisson law of parameter $\lambda$. It is well known that the process $\widetilde{N}$ is a process with independent and stationary increments and that $\widetilde{M}_t \stackrel{def}{=} \widetilde{N}_t - \lambda t$ is a martingale in its canonical filtration (see Brémaud [7] for more details).

The stochastic integral with respect to a Poisson process is easily defined as

$$\int_0^t \psi_s dN_s = \sum_{n, T_n \le t} \psi_{T_n}.$$

More generally, the process $(\widetilde{N}_t, t \ge 0)$ is a Poisson process of intensity $(\widetilde{\lambda}_t, t \ge 0)$ if: (i) $\widetilde{N}$ is a càdlàg process, constant between two jumps, with jumps of size 1, i.e. $\Delta \widetilde{N}_s = \widetilde{N}_s - \widetilde{N}_{s-}$ equals 0 or 1, (ii) the process

$$\widetilde{M}_t \stackrel{def}{=} \widetilde{N}_t - \int_0^t \widetilde{\lambda}_s \, ds$$

is a martingale. Here, $\widetilde{\lambda}$ is a nonnegative process. Then, if the process $\psi$ is bounded and predictable, the process

$$\int_0^t \psi_s dM_s = \int_0^t \psi_s d\widetilde{N}_s - \int_0^t \psi_s \widetilde{\lambda}_s ds = \sum_{n, T_n \le t} \psi_{T_n} - \int_0^t \psi_s \widetilde{\lambda}_s ds$$

is a martingale. In particular,

$$E\Big(\int_0^t \psi_s dN_s\Big) = E\Big(\int_0^t \psi_s \widetilde{\lambda}_s ds\Big).$$

Let $X$ and $Y$ be two semimartingales. The integration by parts formula reads

$$X_t Y_t = X_0 Y_0 + \int_{]0,t]} X_{s-} dY_s + \int_{]0,t]} Y_{s-} dX_s + [X, Y]_t,$$

where $[X, Y]$ is the quadratic covariation (the *bracket*) of the processes $X$ and $Y$. Let us recall that the quadratic covariation of a continuous process and a pure jump process is equal to 0, and that the quadratic covariation of two pure jump processes is the sum of products of their jumps:

$$[\widetilde{N}_1, \widetilde{N}_2]_t = \sum_{s \le t} \Delta(\widetilde{N}_1)_s \Delta(\widetilde{N}_2)_s,$$

where $\Delta(\widetilde{N}_i)_s = (\widetilde{N}_i)_s - (\widetilde{N}_i)_{s-}$. The bracket of a Brownian motion and a Poisson process is equal to 0, and the bracket of $\widetilde{N}$ is $[\widetilde{N}, \widetilde{N}]_t = \widetilde{N}_t$.

# 2 Structural Approach

As already mentioned, there are two basic approaches to modelling default risk. In the first approach – pioneered by Black and Scholes [4] and Merton [44] – the default occurs when the assets of the firm are insufficient to meet payments on debt. If $B$ is the debt value, the payment is $\max(0, V_T - B)$, and thus we essentially deal with the same problem as in the options pricing theory.

In another approach the firm defaults when its value falls below a prespecifed level. In this case, the default time $\tau$ is supposed to be a stopping time in the asset's filtration. The valuation of a defaultable claim reduces here to the problem of the pricing of the claim $X1\!\!1_{\{T<\tau\}}$ which is measurable with respect to the filtration of the prices at time $T$. Main papers to be quoted here are: Briys and de Varenne [8], Ericsson and Reneby [24], Wang [56]. Some authors investigate also the consequences of the renegotiation of the debt: Decamps and Faure-Grimaud [12], Mella-Barral [43].

The valuation of the defaultable claim within the structural approach is a standard (but difficult) problem, which needs the knowledge of the law of the pair $(\tau, X)$. We recall here some of the main mathematical results on this subject.

## 2.1 Hitting Times of a Constant Barrier

Let $V$ be a process starting at $v$. For any $a \geq v$ we introduce the first time where this process reaches $a$, i.e., $\tau_a(V) = \inf\{t \geq 0 \,:\, V_t \geq a\}$. The probability law of $\tau_a(V)$, or at least its Laplace transform, can be explicitly computed in some cases. The value of a defaultable claim on $h(V_T)$ is $E(h(V_T)1\!\!1_{\{T<\tau_a(V)\}})$. Its evaluation requires the knowledge of the probability law of the pair $(V_T, \tau_a(V))$ under the e.m.m. The same studies can be done for $a < v$, if we set

$$\tau_a^-(V) = \inf\{t \geq 0 \,:\, V_t \leq a\} = \tau_{-a}(-V)\,.$$

### 2.1.1 Standard Brownian Motion

Let $V$ be a standard Brownian motion starting at $v$, that is, $V_t = v + W_t$. In this case, the two events $\{\tau_a(V) \leq t\}$ and $\{\sup_{s\leq t} W_s \geq \alpha\}$, where $\alpha = a - v$ are equal. The reflection principle implies that the r.v. $\sup_{s\le t} W_s$ is equal in law to the r.v. $|\, W_t \,|$. Therefore,

$$\begin{aligned} P(\tau_\alpha(W) \le t) &= P(\sup_{s\le t} W_s \ge \alpha) = P(\,|\, W_t \,| \ge \alpha) \\ &= P(W_t^2 \ge \alpha^2) = P(tG^2 \ge \alpha^2) \end{aligned}$$

where $G$ is a Gaussian variable, with mean 0 and variance 1. It follows that $\tau_\alpha(W) \overset{law}{=} \dfrac{\alpha^2}{G^2}$, and the probability density function of $\tau_a(V)$ is

$$f(t) = \frac{a-v}{\sqrt{2\pi t^3}} \exp\Big(-\frac{(v-a)^2}{2t}\Big)\,.$$

The computation of

$$E\big(1\!\!1_{\{T<\tau_a(V)\}} h(V_T)\big) = E\big(h(V_T)\big) - E\big(1\!\!1_{\{T\ge\tau_a(V)\}} h(V_T)\big)$$

can be done using Markov property:

$$\begin{aligned} E\big(1\!\!1_{\{T\ge\tau_a(V)\}} h(V_T)\big) &= E\big(1\!\!1_{\{T\ge\tau_a(V)\}}\, E(h(V_T)\,|\,\mathcal{F}_{\tau_a(V)})\big) \\ &= E\big(1\!\!1_{\{T\ge\tau_a(V)\}}\, h(\widetilde{W}_{T-\tau_a(V)} + a)\big) \\ &= \int_0^T du\, f(u) E\big(h(\widetilde{W}_{T-u} + a)\big)\,, \end{aligned}$$

where $\widetilde{W}$ is a Brownian motion independent of $\tau_a(V)$, with $\widetilde{W}_0 = 0$.

### 2.1.2 Brownian Motion with Drift

Suppose that $V_t = v + \mu t + \sigma W_t$, with $\sigma > 0$. Then, $\{\tau_a(V) \le t\} = \{\tau_\alpha(\widetilde{V}) \le t\}$, where $\widetilde{V}_t = \dfrac{\mu t}{\sigma} + W_t$ and $\alpha = (a-v)/\sigma$. From Girsanov's theorem, we can deduce the law of $\tau_a(V)$. Indeed, if we denote for simplicity $\tau_\alpha = \tau_\alpha(W)$, then

$$\begin{aligned} P(\tau_\alpha(\widetilde{V}) \ge t) &= E\Big(1\!\!1_{\{\tau_\alpha \ge t\}} \exp\big(\frac{\mu}{\sigma} W_{\tau_\alpha} - \frac{1}{2}\frac{\mu^2}{\sigma^2}\tau_\alpha\big)\Big) \\ &= \exp\big(\frac{\mu}{\sigma}\alpha\big)\, E\Big(1\!\!1_{\{\tau_\alpha \ge t\}} \exp\big(-\frac{1}{2}\frac{\mu^2}{\sigma^2}\tau_\alpha\big)\Big) \end{aligned}$$

and the quantity

$$E\Big(1\!\!1_{\{\tau_\alpha \geq t\}} \exp\big(-\frac{1}{2}\frac{\mu^2}{\sigma^2}\tau_\alpha\big)\Big)$$

can be computed from the density of $\tau_\alpha$. Indeed, tedious computations lead to

$$E(1\!\!1_{\{\tau_\alpha < s\}}\, e^{-\frac{\nu^2}{2}\tau_\alpha}) = H(\nu, |\,\alpha\,|, s)$$

where

$$H(\nu, x, s) = e^{-\nu x}\mathcal{N}\big(\nu - \frac{x}{\sqrt{s}}\big) + e^{\nu x}\mathcal{N}\big(-\nu - \frac{x}{\sqrt{s}}\big) \qquad (1)$$

and $\mathcal{N}$ stands for the cumulative distribution function of the standard Gaussian law. The density of $\tau_\alpha(\widetilde{V})$ is

$$P_0(\tau_\alpha(\widetilde{V}) \in dt) = \frac{|\alpha|}{\sqrt{2\pi t^3}} \exp -\frac{(\alpha - \tilde{\mu}t)^2}{2t}$$

where $\tilde{\mu} = \mu/\sigma$ (see Borodin and Salminen [5] p. 223, formula 2.0.2.)

### 2.1.3 Geometric Brownian Motion

If $V$ is a geometric Brownian motion such that

$$dV_t = V_t(\mu\, dt + \sigma\, dW_t)$$

then for $a > v > 0$ we have

$$\begin{aligned} \tau_a(V) &= \inf\big\{t \geq 0 \,:\, v \exp\big((\mu - \frac{\sigma^2}{2})t + \sigma W_t\big) \geq a\big\} \\ &= \inf\big\{t \geq 0 \,:\, \big(\mu - \frac{\sigma^2}{2}\big)t + \sigma W_t \geq \ln\frac{a}{v}\big\} \end{aligned}$$

so that the problem reduces to the case of a Brownian motion with drift.

### 2.1.4 Deterministic Volatility

If $dV_t = \sigma(t)\, dW_t$, where $W$ is a Brownian motion and $\sigma$ is a deterministic function, a change of time will give the answer. In fact, from the Dambis-Dubins-Schwarz theorem, the martingale $V$ is a time-changed Brownian motion. More precisely, there exists a Brownian motion $\widetilde{W}$

such that $V_t = \widetilde{W}_{A(t)}$ where $A(t) = \langle V, V\rangle_t = \int_0^t \sigma^2(s)\,ds$. Using this change of time, and the equality

$$\{\tau_a(V) < t\} = \{\sup_{s\le t} V_s > a\} = \{\sup_{s\le t} \widetilde{W}_{A(s)} > a\} = \{\tau_a(\widetilde{W}) < A(t)\}$$

we get the result. To the best of our knowledge, no closed-form solution is known for the probability law of the hitting time when $V$ satisfies $dV_t = \mu\,dt + \sigma(t)\,dW_t$.

### 2.1.5 Ornstein-Uhlenbeck Process

Let $(r_t, t \ge 0)$ be defined as

$$dr_t = (\phi - \lambda r_t)\,dt + \sqrt{\beta}\,dW_t, \quad r_0 = r,$$

and $\tau_\rho = \inf\{t \ge 0 \,:\, r_t \ge \rho\}$. For any $\rho > r$, the density function of $\tau_\rho$ equals

$$\begin{aligned} f(t) = \frac{\rho - r_0}{\sqrt{2\beta\pi t^3}} \left(\frac{\lambda t}{\sinh \lambda t}\right)^{3/2} e^{\lambda t/2} \\ \exp\Big[-\frac{\lambda}{2\beta}\Big(\big(\rho - \frac{\phi}{\lambda}\big)^2 - \big(r_0 - \frac{\phi}{\lambda}\big)^2 + (\rho - r_0)^2 \coth \lambda t\Big)\Big]. \end{aligned}$$

For the derivation of the last formula, the reader is referred to Leblanc's thesis [37] (where there are some misprints in the result, however).

### 2.1.6 Bessel Processes

A Bessel process $R$ with index $\nu \ge 0$ (or with dimension $\delta$ with $\nu = \dfrac{\delta}{2} - 1$) is a diffusion process which takes values in $\mathbb{R}_+$, and has the infinitesimal generator

$$\mathcal{A}^\nu = \frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu + 1}{2x}\frac{d}{dx}.$$

For $\delta > 1$, a $BES(\delta)$ satisfies $E\big(\int_0^t R_s^{-1} ds\big) < \infty$, and it is the solution of

$$R_t = \alpha + W_t + \frac{\delta - 1}{2}\int_0^t \frac{1}{R_s}\,ds\,. \qquad (2)$$

In terms of the index $\nu$

$$R_t = \alpha + W_t + \big(\nu + \frac{1}{2}\big)\int_0^t \frac{1}{R_s}\,ds\,.$$

It is possible to derive a closed-form expression for the Laplace transform of the probability law of the hitting time. For example, if $V$ is a $BES(3)$ process then for any $0 < v < y$ (see [5])

$$E_v^{(3)}\big(\exp - \frac{\lambda^2}{2}\tau_y\big) = \frac{y\,\sinh(\lambda v)}{v\,\sinh(\lambda y)}.$$

For a $BES$ with index $\nu$, the Laplace transform is given via Bessel modified functions, namely,

$$E_v^{\nu}\big(\exp - \frac{\lambda^2}{2}\tau_y\big) = \left(\frac{y}{v}\right)^{\nu}\frac{I_\nu(\lambda v)}{I_\nu(\lambda y)}\,.$$

For a $BES(3)$, it is also possible to find the density of the hitting time. The absolute continuity relationship

$$P_x^{(3)}\big|_{\mathcal{F}_t} = \frac{X_{t\wedge\tau_0}}{x}\,P_x\big|_{\mathcal{F}_t},$$

where $P_x$ is the law of Brownian motion started at $x$ and $X$ the canonical process on the Wiener space, yields the equality (see Revuz and Yor [47])

$$P_x^{(3)}(\phi(\tau_a)\mathbb{1}_{\{\tau_a<\infty\}}) = \frac{a}{x}P_x(\phi(\tau_a))$$

which holds for $a < x$. Consequently (as before, $G$ stands for the r.v. with the standard Gaussian law under $P$)

$$\begin{aligned}
P_x^{(3)}(\tau_a > t) &= P_x^{(3)}(\infty > \tau_a > t) + P_x^{(3)}(\tau_a = \infty) \\
&= \frac{a}{x}P_0(\tau_{x-a} > t) + (1 - \frac{a}{x}) \\
&= \frac{a}{x}P(x - a > \sqrt{t}\,|\,G\,|\,) + (1 - \frac{a}{x}) \\
&= \frac{a}{x}\sqrt{\frac{2}{\pi}}\int_0^{(x-a)/\sqrt{t}} e^{-y^2/2}\,dy + (1 - \frac{a}{x}).
\end{aligned}$$

For $a > x$, the density of $\tau_a$ involves a series (see Borodin and Salminen [5], p. 339, formula 2.0.2 and p. 387, formula 2.0.2.)

### 2.1.7 Time-homogeneous Diffusion

Let $V$ be a diffusion of the form

$$dV_t = a(V_t)\,dt + \sigma(V_t)\,dW_t,$$

where $a$ and $\sigma$ are Lipschitz continuous functions. Let $\varphi$ be a bounded function on $]-\infty, a[$ such that $\mathcal{A}\varphi = \lambda\varphi$, where $\mathcal{A}$ is the infinitesimal generator of $V$, that is,

$$\mathcal{A} = \frac{1}{2}\sigma^2(x)\,\frac{d^2}{dx^2} + a(x)\,\frac{d}{dx}\,.$$

Then

$$E_v\left(1\!\!1_{\{\tau_a<\infty\}}e^{-\lambda\tau_a}\right) = \frac{\varphi(v)}{\varphi(a)}$$

for any $v \in ]-\infty, a[$ and $\lambda > 0$.

### 2.1.8 Non-constant Barrier

Let $\tau_f(V) = \inf\{t \geq 0 : V_t = f(t)\}$, where $f$ is a deterministic function and $V$ a diffusion process. There are only few cases for which the law of $\tau_f(V)$ is explicitly known; for instance, the case when $V$ is a Brownian motion and $f$ is an affine function.

## 2.2 Stochastic Barrier

In this section (taken from [23]) we study a simple example which introduce the main tools of all our study: the choice of filtration and the conditional probability distribution function.

- **Filtrations**

Suppose that a space $(\Omega, \mathcal{G}, \mathbf{F}, P)$ is given, where $\mathbf{F} = (\mathcal{F}_t)_{\{t\geq 0\}}$ is a filtration such that $\mathcal{F}_\infty \subset \mathcal{G}$. We define the default time $\tau$ as $\tau = \inf\{t \geq 0 : V_t < H\}$, where $H$ is a random variable and $V$ an $\mathbf{F}$-adapted process, solution of a SDE driven by a Brownian motion $W$. We can write

$$\tau = \inf\{t \geq 0 : V_t^* > \Theta\},$$

where $V^*$ is an increasing process, namely,

$$V_t^* \stackrel{def}{=} \sup\{-(V_s - V_0), s \le t\}$$

and $\Theta = V_0 - H$. We assume that $\Theta > 0$. In this section, we assume that the random variable $\Theta$ is known, and that the information available in the market at time $t$ is $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\Theta)$. Then, obviously, $\{\tau \le t\} \in \mathcal{G}_t$ and $\tau$ is a $\mathbf{G}$-stopping time. Suppose that $\mathbf{F}$ is a Brownian filtration and that the $\mathbf{F}$-Brownian motion $W$ remains a $\mathbf{G}$-Brownian motion. Then $W$ is independent of $\mathcal{G}_0$ and thus it is independent of $\Theta$. The dynamics of the assets remain the same for the two filtrations $\mathbf{F}$ and $\mathbf{G}$ only if $H$ is independent of the filtration $\mathbf{F}$.

- **Conditional law**

Here we assume, as in El Karoui [23], that the barrier is independent of $\mathbf{F}$, and we introduce the distribution function of $\Theta$: $F(t) = P(\Theta \le t)$. We assume that $F$ is continuous and that $F(0) = 0$. Let us introduce the function $K$ defined as $1 - F(t) = P(\Theta > t) \stackrel{def}{=} e^{-K(t)}$. The function $K$ is an increasing function, such that $K(0) = 0$ and $K(\infty) = \infty$.

The conditional law of $\tau$ with respect to $\mathcal{F}_\infty$ is

$$P(\tau \le t \,|\, \mathcal{F}_\infty) = P(V_t^* \ge \Theta \,|\, \mathcal{F}_\infty) = F(V_t^*) = 1 - e^{-K(V_t^*)}.$$

In particular, if $V_\infty^* \ge \Theta$ a.s., then $\tau$ is finite with probability 1. Furthermore, in the particular case when $\Theta$ has an exponential law with parameter 1, we have $P(\tau > t \,|\, \mathcal{F}_\infty) = \exp(-V_t^*)$.

If $V_t^*$ is absolutely continuous with respect to the Lebesgue measure, the process $\lambda$ such that $V_t^* = \int_0^t \lambda_s \, ds$ satisfies

$$\lambda_t = \lim_{h \to 0} \frac{P(t < \tau \le t + h \,|\, \mathcal{F}_t)}{P(\tau > t \,|\, \mathcal{F}_t)}.$$

Let us now focus on the converse implication. Suppose that

$$P(\tau \le t \,|\, \mathcal{F}_\infty) = e^{-K(A_t)}$$

where $A$ is an arbitrary continuous increasing $\mathbf{F}$-adapted process, and $K$ a continuous increasing function. Our goal is to show that there exists a random variable $\Theta$, which is independent of $\mathcal{F}_\infty$, and such that $\tau \overset{law}{=} \inf\{t \geq 0 \,:\, \Lambda_t > \Theta\}$. Let us set $\Theta \overset{def}{=} \Lambda_\tau$. Then

$$\{t < \Theta\} = \{t < \Lambda_\tau\} = \{C_t < \tau\},$$

where $C$ is the right inverse of $\Lambda$, so that $\Lambda_{C_t} = t$. Therefore

$$P(\Theta > t \,|\, \mathcal{F}_\infty) = e^{-K(\Lambda_{C_t})} = e^{-K(t)}.$$

We have thus established the required properties, namely, the probability law of $\Theta$ and its independence of the $\sigma$-field $\mathcal{F}_\infty$.

## 2.3 Jump-diffusion

Zhou [60] studies the case when the value of the firm is modelled via a jump-diffusion process on the form

$$dV_t = V_{t-}\big((\mu - \lambda\nu)\,dt + \sigma\,dW_t + \Phi_t\,dN_t^*\big), \quad V_0 = v,$$

where $W$ is a Brownian motion, $N^*$ is a Poisson process, and $\Phi$ represents the jump amplitude. The processes $W, N^*$ and $\Phi$ are assumed to be mutually independent. The default time is modelled as the first time when the process $V$ falls below the level $a$. Zhou computes the expectation of the discounted payoff under the risk neutral measure, assuming that the risk premium associated with the jump is equal to zero. A closed-form expression for the probability law of the pair $(V_T, \tau)$ is not known in this case. The difficulty is that the level may be crossed either in a continuous way or with a jump. Zhou gives an approximation of the expectation, using a certain time discretization of the process $V$.

If $\Phi$ is a nonnegative constant and $v > a$, we have $V_\tau = a$, and the Laplace transform of $\tau$ can be easily found.

## 2.4 Hedging

The defaultable contingent claim $\mathbb{1}_{\{T<\tau\}} h(V_T)$ is an $\mathcal{F}_T$-measurable random variable. If the default-free market is complete, this remains true for the defaultable market and it is possible to hedge any defaultable contingent claim. This is not the case in Zhou's model, however.

### 2.5 Term Structure Models

A substantial literature proposes to model both the default free term structure and the term structure representing the relative prices of different maturities of default-risky debt, using an extension of the method developed by Heath-Jarrow-Morton. Major papers in this area include Jarrow and Turnbull [29], Schönbucher [54, 52], Hubner [27, 26], and Bielecki and Rutkowski [3].

## 3 Intensity-based Approach

From now on, we shall focus our attention on the intensity-based valuation. The default time $\tau$ is given as a random time, i.e., a nonnegative random variable. We associate with this random time the counting process $N$ defined as $N_t = \mathbb{1}_{\{\tau \le t\}}$. The process $N$ is an increasing process, càdlàg, equal to 0 before the default and equal to 1 after default. Essentially, the *intensity* of $\tau$ is defined as the nonnegative adapted process $\lambda$ such that

$$M_t \stackrel{def}{=} N_t - \int_0^{t\wedge\tau} \lambda_u \, du$$

is a martingale. This approach, more recent than the structural one, is also known as the *reduced-form approach,* and has been introduced by Jarrow and Turnbull [29], Jarrow, Lando and Turnbull [30], Lando [34, 35], Duffie and Singleton [17]. More recents contributions are Hubner [27, 26] Arvanatis, Gregory and Laurent [1], Schönbucher [51, 52, 54, 53] and Lotz [39, 40] among others.

As we shall see in what follows, the choice of the filtration is essential.

### 3.1 Stochastic Intensity: Classic Approach

In this section, we adopt the standard definition of stochastic intensity of a random time $\tau$ with respect to a filtration $\mathbf{J}$ such that $\tau$ is a $\mathbf{J}$-stopping time. Namely, we say that, for a given filtration $\mathbf{J} = (\mathcal{J}_t)_{\{t\ge 0\}}$, the $\mathbf{J}$-adapted nonnegative process $\lambda$ is the $\mathbf{J}$-*intensity* (or briefly, the *stochastic intensity*) of $\tau$ if the process $N_t - \int_0^{\tau\wedge t} \lambda_s \, ds$ is a $\mathbf{J}$-martingale. We emphasize that if such a definition of a stochastic intensity of $\tau$ is adopted, then the stochastic intensity $\lambda$ has virtually no meaning after time $\tau$.

Let us consider an elementary example. If $\tau$ is an exponentially distributed random variable on some filtered probability space $(\Omega, \mathbf{J}, P)$, with the parameter $\lambda > 0$, then the constant $\lambda$ is referred to as the *hazard rate* of $\tau$, and the *stochastic intensity* of $\tau$ (with respect to any filtration such that $\tau$ is a $\mathbf{J}$-stopping time) equals $\lambda_t = \lambda \mathbb{1}_{\{t \leq \tau\}}$. Such a definition of stochastic intensity is quite sufficient in the theory of (marked) point processes[2] since in this case one is interested mainly in the filtration generated by the point process itself.

In financial applications, however, we frequently deal with some prespecified underlying filtration,[3] $\mathbf{F}$ say, and thus it is more appropriate to introduce the more specific concept of a $\mathbf{F}$-*intensity* of a random time. The case of a $\mathbf{F}$-intensity with respect to an external filtration $\mathbf{F}$ is examined in Section 5 below (see, in particular, Section 5.1.6). In fact, we find it convenient to introduce two related notions: of a $\mathbf{F}$-*hazard process* $\Gamma$ and a $\mathbf{F}$-*martingale hazard process* $\Lambda$ of a random time $\tau$ (see Definitions 5.1 and 5.2, respectively). One of our main goals is to study the relationships between these two concepts, under various types of hypotheses imposed on the underlying filtrations.

**Remark 3.1** Let $\widehat{\mathbf{J}}$ be a filtration larger than $\mathbf{J}$, i.e., $\mathcal{J}_t \subset \widehat{\mathcal{J}}_t$ for every $t \geq 0$, and $\lambda$ the $\mathbf{J}$-intensity of $N$. The process $N$ is manifestly $\widehat{\mathbf{J}}$ adapted, and thus $\tau$ is still a stopping time with respect to the enlarged filtration $\widehat{J}$. However, it may happen that $N$ does not admit an intensity in the filtration $\widehat{\mathbf{J}}$, and if the $\widehat{\mathbf{J}}$-intensity exists, it may be different from $\mathbf{J}$ intensity.

In the remaining part of this section, we present the main results and examples which can be found in existing financial literature.

### 3.1.1 Conditional Expectation

Suppose that $\tau$ is a random time with $\mathbf{J}$-intensity $\lambda$ – that is, the process $M_t = N_t - \int_0^{t \wedge \tau} \lambda_s ds$ is a $\mathbf{J}$-martingale. Let $\mathbf{F}$ be a subfiltration of $\mathbf{J}$.

[2]See, for instance, Last and Brandt [36].

[3]Typically, it is generated by the observations of price processes of primary assets.

**Definition 3.1** The pair $(h, X)$ where $h$ is a $\mathbf{F}$-predictable process and $X$ a $\mathcal{F}_T$-measurable nonnegative random variable is called a $\mathbf{F}$-*adapted defaultable claim.* It corresponds to a terminal payoff $X$ which is paid if the default has not appeared before or at time $T$, and a rebate $h$ which is paid when the default appears.

Let $(h, X)$ be a $\mathbf{F}$-adapted defaultable claim with the *value process* $S$. Let

$$R_t S_t = E\big(R_\tau h_\tau \mathbb{1}_{\{t<\tau\le T\}} + X R_T \mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{J}_t\big)$$

be its discounted value. We assume here that the *savings account* $B$ satisfies

$$B_t = \exp\Big(\int_0^t r_u\,du\Big)$$

for some $\mathbf{F}$-adapted nonnegative process $r$ (known as the *short-term interest rate*), and the *discount factor* $R$ equals $R_t = B_t^{-1}$.

**Proposition 3.1** *Let* $\widehat{\lambda}_u = \lambda_u(1 - N_u)$. *Then*

$$R_t S_t = E\Big(\int_t^T R_u h_u \widehat{\lambda}_u du + X R_T \mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{J}_t\Big) \tag{3}$$

*and*

$$R_t S_t = E\Big(\int_t^T (h_u \widehat{\lambda}_u - r_u S_u) du + X R_T \mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{J}_t\Big). \tag{4}$$

PROOF: From the definition of the stochastic integral, we get

$$R_\tau h_\tau \mathbb{1}_{\{t<\tau\le T\}} = \int_t^T R_u h_u dN_u = \int_t^T R_u h_u dM_u + \int_t^T R_u h_u \widehat{\lambda}_u du.$$

Then, the martingale property of the stochastic integral with respect to $M$ yields

$$\begin{aligned} E\big(R_\tau h_\tau \mathbb{1}_{\{t<\tau\le T\}} \,|\, \mathcal{J}_t\big) &= E\Big(\int_t^T R_u h_u dN_u \,|\, \mathcal{J}_t\Big) \\ &= E\Big(\int_t^T R_u h_u \widehat{\lambda}_u du \,|\, \mathcal{J}_t\Big). \end{aligned}$$

The formula above proves that the process $R_t S_t + \int_0^t R_u \widehat{\lambda}_u h_u du$ is a $\mathbf{J}$-martingale. This immediately yields (3). Furthermore, using Itô's formula we conclude that the process $S_t + \int_0^t (\widehat{\lambda}_u h_u - r_u S_u) du$ is a $\mathbf{J}$-martingale and this proves (4). □

### 3.1.2 Hypothesis (D)

As before, $\tau$ is a random time with $\mathbf{J}$-intensity $\lambda$, i.e., the process $M_t = N_t - \int_0^{t\wedge\tau} \lambda_s ds$ is a $\mathbf{J}$-martingale. In order to give the value of a defaultable claim in a neat form, many authors (see, e.g., Duffie et al. [19, 20]) make the following hypothesis:

**Hypothesis (D).** The intensity $\lambda$ admits at least one extension up to infinity, say $\lambda^*$ such that such that the (right-continuous) process $V$, given by the formula

$$V_t = E\big(Ye^{-\int_t^T \lambda_u^*\,du} \mid \mathcal{J}_t\big), \tag{5}$$

is continuous at $\tau$, that is, $\Delta V_{T\wedge\tau} = V_{T\wedge\tau} - V_{(T\wedge\tau)-} = 0$.

Their main result is the following:

**Proposition 3.2** *For a fixed $T > 0$, let $Y$ be a $\mathcal{J}_T$-measurable random variable. Under hypothesis* (D) *we have, for any $t < T$,*

$$E(\mathbb{1}_{\{\tau>T\}}Y \mid \mathcal{J}_t) = \mathbb{1}_{\{\tau>t\}}\, E\big(Ye^{-\int_t^T \lambda_u^*\,du} \mid \mathcal{J}_t\big). \tag{6}$$

PROOF: We shall first check that

$$\mathbb{1}_{\{\tau>t\}}V_t = E\big(\Delta V_\tau \mathbb{1}_{\{t<\tau\le T\}} + \mathbb{1}_{\{\tau>T\}}Y \,\big|\, \mathcal{J}_t\big). \tag{7}$$

From the definition, $V_t = e^{\Lambda_t^*}\widetilde{M}_t$, where $\widetilde{M}$ is a $\mathbf{J}$-martingale: $\widetilde{M}_t = E\big(Ye^{-\Lambda_T^*} \mid \mathcal{J}_t\big)$ for $t \in [0,T]$, and $\Lambda_t^* = \int_0^t \lambda_s^*\,ds$. Using Itô's product rule, we obtain

$$dV_t = \widetilde{M}_{t-}\,d\,(e^{\Lambda_t^*}) + e^{\Lambda_t^*}\,d\widetilde{M}_t = V_{t-}e^{-\Lambda_t^*}\,d\,(e^{\Lambda_t^*}) + e^{\Lambda_t^*}\,d\widetilde{M}_t. \tag{8}$$

Define $U_t = \widehat{N}_t V_t$, where $\widehat{N}_t = \mathbb{1}_{\{\tau>t\}} = 1 - N_t$, and observe that (7) may be rewritten as follows

$$U_t = E\Big(\int_{]t,T]} \Delta V_u\,dN_u + \mathbb{1}_{\{\tau>T\}}Y \,\Big|\, \mathcal{J}_t\Big). \tag{9}$$

On the other hand, an application of Itô's product rule yields

$$dU_t = \widehat{N}_{t-}\,dV_t - V_{t-}\,dN_t + \Delta V_t \Delta\widehat{N}_t.$$

Combining the last formula with (8), and noticing that $\Delta\widehat{N}_t = -\Delta N_t$, we obtain

$$dU_t = \widehat{N}_{t-}\big(V_{t-}e^{-\Lambda^*_t}\,d(e^{\Lambda^*_t}) + e^{\Lambda^*_t}\,d\widetilde{M}_t\big) - V_{t-}\,dN_t - \Delta V_t\,dN_t.$$

After rearranging, we get

$$dU_t = -\Delta V_t\,dN_t + d\widehat{M}_t, \tag{10}$$

where $\widehat{M}$ stands for a $\mathbf{J}$-martingale. More precisely,

$$d\widehat{M}_t = \widehat{N}_{t-}e^{\Lambda^*_t}\,d\widetilde{M}_t + dM^*_t,$$

where

$$dM^*_t = -V_{t-}\big(dN_t - 1\!\!1_{\{\tau\geq t\}}e^{-\Lambda^*_t}\,d(e^{\Lambda^*_t})\big) = -V_{t-}\,d(N_t - \Lambda_{t\wedge\tau}),$$

so that $M^*$ is a $\mathbf{J}$-martingale. Since obviously $U_T = 1\!\!1_{\{\tau>T\}}Y$, (10) implies (9). If $V$ is continuous at $\tau$ then (7) yields

$$\begin{aligned} E(1\!\!1_{\{\tau>T\}}Y\,|\,\mathcal{J}_t) &= 1\!\!1_{\{\tau>t\}}E(1\!\!1_{\{\tau>T\}}Y\,|\,\mathcal{J}_t) = 1\!\!1_{\{\tau>t\}}V_t \\ &= 1\!\!1_{\{\tau>t\}}E\big(Ye^{\Lambda^*_t-\Lambda^*_T}\,|\,\mathcal{J}_t\big). \end{aligned}$$

This completes the proof. □

**Remark 3.2** It should be observed that the 'natural' extension $\lambda^*_u = \lambda_u 1\!\!1_{]0,\tau]}$ does not satisfy (D). Indeed, it can be shown that the process $E\big(e^{-\int_t^\tau \lambda^*_u\,du}\,|\,\mathcal{J}_t\big)$ is discontinuous at $\tau$. Moreover, this hypothesis can not be satisfied for every $Y$. Indeed, it would imply that every $\mathbf{J}$-martingale is continuous, but this is manifestly not true. Therefore, the suitable choice of the extended intensity process $\lambda^*$ should depend on $Y$.

## 3.2 Cox Processes and Extensions

Let $(\Omega, \mathbf{F}, P)$ be a filtered probability space. An example of default time with stochastic intensity is a single jump Cox process, that is, a process $N_t = 1\!\!1_{\{t\leq\tau\}}$ such that there exists an $\mathbf{F}$-adapted process $f$ with

$$P(\tau\leq t\,|\,\mathcal{F}_\infty) = \int_0^t f_s\,ds \overset{def}{=} F_t.$$

In this case

$$P(\tau \leq t \,|\, \mathcal{F}_t) = F_t = 1 - \exp\Big( - \int_0^t \lambda_u \, du \Big),$$

where $\lambda_s = \dfrac{f_s}{1 - F_s}$. In order to study the intensity of $\tau$, we have to introduce a filtration such that $\tau$ is a stopping time; this is not the case for $\mathbf{F}$. Indeed, this would imply that $F_t = P(\tau \leq t \,|\, \mathcal{F}_t)$ is equal to $N_t$, which is manifestly not true.

### 3.2.1 Elementary Case

In the early papers on intensity-based approach to credit risk modelling (see, for instance, [29, 34]) the default time is modelled as the first jump of a Poisson process, which is assumed to be independent of the assets prices and/or of the value of the firm.

Suppose that $\mathbf{F}$ is the filtration of the assets prices and that $\widetilde{N}$ is a Poisson process, independent of $\mathbf{F}$, with stochastic intensity $\widetilde{\lambda}$. We denote by $\tau$ the first time when the Poisson process has a jump, i.e., $\tau = T_1$. By $N_t = \widetilde{N}_{t \wedge \tau}$ we denote the associated single jump process. The canonical filtration of $N$ is $\mathbf{H} = (\mathcal{H}_t, t \geq 0)$, where $\mathcal{H}_t = \sigma(N_s, s \leq t)$, and the intensity is clearly $\mathbf{H}$-adapted. Finally, $\mathcal{G}_t$ stands for the $\sigma$-algebra generated by $\mathcal{F}_t$ and $\mathcal{H}_t$, we write briefly: $\mathbf{G} = \mathbf{H} \vee \mathbf{F}$.

The $\mathbf{H}$-martingale $M$, stopped at the $\mathbf{H}$-stopping time $\tau$

$$M_t = \widetilde{M}_{t \wedge \tau} = N_t - \int_0^{t \wedge \tau} \widetilde{\lambda}_s ds$$

is a $\mathbf{H}$-martingale. We denote by $\lambda$ a stochastic process which is equal to $\widetilde{\lambda}$ up to time $\tau$.

It is useful to observe that the process $(M_t = N_t - \int_0^{t \wedge \tau} \lambda_s ds \,,\, t \geq 0)$ is not only a $\mathbf{H}$-martingale, but also a $\mathbf{G}$-martingale.[4] The independence property allows us to state that any bounded $\mathcal{F}_\infty$-measurable r.v. $X$ we have

$$E(X \mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \, E\Big( \exp - \int_t^T \lambda_u \, du \,\Big|\, \mathcal{H}_t \Big) \, E(X \,|\, \mathcal{F}_t).$$

[4]Notice that a martingale in a given filtration is not necessarily a martingale in a larger filtration.

Indeed, it suffices to recall that, if **F** and **H** are independent filtrations then for any bounded $\mathcal{F}_\infty$-measurable r.v. $X$ and any bounded $\mathcal{H}_\infty$-measurable r.v. $Y$, we have $E(XY \,|\, \mathcal{F}_t \vee \mathcal{H}_t) = E(X \,|\, \mathcal{F}_t)\, E(Y \,|\, \mathcal{H}_t)$ for any $t$. Such a model is used in the literature on credit ratings, where the default-time is represented by the first time where a Markov-chain reaches an absorbing state (see [35]).

### 3.2.2 Construction of Cox Processes

Let $(\Omega, \mathcal{G}, P)$ be a probability space, and $(X_t, t \geq 0)$ a continuous diffusion process on this space. We denote by **F** its canonical filtration, satisfying the usual conditions. A nonnegative function $\lambda$ is given. We assume that there exists a random variable $\Theta$, independent of $X$, with an exponential law: $P(\Theta \geq t) = e^{-t}$. We define the random time $\tau$ as the first time when the process $\int_0^t \lambda(X_s)\, ds$ is above the random level $\Theta$, i.e.,

$$\tau = \inf \{ t \geq 0 \,:\, \int_0^t \lambda(X_s)\, ds \geq \Theta \}.$$

The mutual independence of $\Theta$ and $X$ will avoid us to enter in the enlargement of filtration's world. This will be done in a next section, in the general case, that is, when the independence hypothesis is relaxed.

Another example is to choose $\tau = \inf \{ t \geq 0 \,:\, \widetilde{N}_{\Lambda_t} = 1 \}$, where $\Lambda_t = \int_0^t \lambda_s \, ds$ and $\widetilde{N}$ is a Poisson process with intensity 1, independent of the filtration **F**. The second method is in fact equivalent to the first. Cox processes are used in a great number of studies (see, e.g., [35, 45]). We shall generalize this approach in Section 7.

### 3.2.3 Conditional Expectations

Let us check that the above process $N$ is a Cox processes.

**Lemma 3.1** *The conditional distribution function of $\tau$ given the $\sigma$-field $\mathcal{F}_t$ is for $t \geq s$*

$$P(\tau > s \,|\, \mathcal{F}_t) = \exp\Big( - \int_0^s \lambda(X_u)\, du \Big).$$

PROOF: The proof follows from the equality $\{\tau > s\} = \{\int_0^s \lambda(X_u)\, du > \Theta\}$. From the independence assumption and the $\mathcal{F}_t$-measurability of $\int_0^s \lambda(X_u)\, du$ for $s \le t$, we obtain

$$P(\tau > s \,|\, \mathcal{F}_t) = P\Big(\int_0^s \lambda(X_u) du \ge \Theta \,\Big|\, \mathcal{F}_t\Big) = \exp\Big(-\int_0^s \lambda(X_u)\, du\Big).$$

In particular, we have

$$P(\tau \le t \,|\, \mathcal{F}_t) = P(\tau \le t \,|\, \mathcal{F}_\infty).$$

Let us notice that the process $F_t = P(\tau \le t \,|\, \mathcal{F}_t)$ is here an increasing process. □

We write $N_t = 1\!\!1_{\{\tau \le t\}}$ and $\mathcal{H}_t = \sigma(N_s \,:\, s \le t)$. We introduce the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, that is, the enlarged filtration generated by the underlying filtration $\mathbf{F}$ and the process $N$. (We denote by $\mathbf{F}$ the original **F**iltration and by $\mathbf{G}$ the enlar**G**ed one.) We shall frequently write $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$.

It is easy to describe the events which belong to the $\sigma$-field $\mathcal{G}_t$ on the set $\{\tau > t\}$. Indeed, if $G_t \in \mathcal{G}_t$, then $G_t \cap \{\tau > t\} = B_t \cap \{\tau > t\}$ for some event $B_t \in \mathcal{F}_t$.

Therefore any $\mathcal{G}_t$-measurable random variable $Y_t$ satisfies $1\!\!1_{\{\tau > t\}} Y_t = 1\!\!1_{\{\tau > t\}} y_t$, where $y_t$ is an $\mathcal{F}_t$-measurable random variable.

We emphasize that the enlarged filtration $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$ is here the filtration which should be taken into account; the filtration generated by $\mathcal{F}_t$ and $\sigma(\Theta)$ is too large. In the latter filtration $\tau$ would be a predictable stopping time, and would not admit an intensity.

**Proposition 3.3** *Let $X$ be an integrable r.v. Then,*

$$1\!\!1_{\{\tau > t\}} E(X \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} \frac{E(X 1\!\!1_{\{\tau > t\}} \,|\, \mathcal{F}_t)}{E(1\!\!1_{\{\tau > t\}} \,|\, \mathcal{F}_t)}.$$

PROOF: From the remarks on the $\mathcal{G}_t$ measurability, if $Y_t = E(X \,|\, \mathcal{G}_t)$, then there exists $y_t$, which is $\mathcal{F}_t$-measurable such that

$$1\!\!1_{\{\tau > t\}} E(X \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} y_t$$

and multiplying both members by the indicator function, we deduce $y_t = \dfrac{E(X 1\!\!1_{\{\tau > t\}} \,|\, \mathcal{F}_t)}{E(1\!\!1_{\{\tau > t\}} \,|\, \mathcal{F}_t)}$. □

We shall now compute the expectation of a predictable process at time $\tau$ and we shall give the intensity of $\tau$.

**Lemma 3.2** (i) *If $h$ is a* $\mathbf{F}$*-predictable process then*

$$\begin{aligned} E(h_\tau) &= E\Big(\int_0^\infty h_u \lambda(X_u) \exp\big(-\Lambda_u\big)\, du\Big) \\ E(h_\tau \,|\, \mathcal{F}_t) &= E\Big(\int_0^\infty h_u \lambda(X_u) \exp\big(-\Lambda_u\big)\, du \,\Big|\, \mathcal{F}_t\Big) \end{aligned}$$

*and*

$$E(h_\tau \,|\, \mathcal{G}_t) = E\Big(\int_t^\infty h_u \lambda(X_u) e^{\Lambda_t - \Lambda_u}\, du \Big| \mathcal{F}_t\Big) 1\!\!1_{\{\tau > t\}} + h_\tau 1\!\!1_{\{\tau \le t\}}. \quad (11)$$

*where* $\Lambda_u = \int_0^u \lambda(X_s) ds$.
(ii) *The process* $(N_t - \int_0^{t\wedge\tau} \lambda(X_s) ds, t \ge 0)$ *is a* $\mathbf{G}$*-martingale.*

Proof: Let $h_t = 1\!\!1_{]v,w]}(t) B_v$ where $B_v \in \mathcal{F}_v$. Then,

$$\begin{aligned} E(h_\tau \,|\, \mathcal{F}_t) &= E\Big(E(1\!\!1_{]v,w]}(\tau) B_v \,|\, \mathcal{F}_\infty) \,\Big|\, \mathcal{F}_t\Big) \\ &= E\Big(B_v\big(e^{-\Lambda_v} - e^{-\Lambda_w}\big) \,|\, \mathcal{F}_t\Big) \\ &= E\Big(B_v \int_v^w \lambda(X_u) e^{-\Lambda_u}\, du \,\Big|\, \mathcal{F}_t\Big) \\ &= E\Big(\int_0^\infty h_u \lambda(X_u) e^{-\Lambda_u}\, du \,\Big|\, \mathcal{F}_t\Big) \end{aligned}$$

and the result follows from the monotone class theorem.

The martingale property (ii) follows from integration by parts formula. Let $t < s$. Then, on the one hand

$$\begin{aligned} E(N_s - N_t \,|\, \mathcal{G}_t) &= P(t < \tau \le s \,|\, \mathcal{G}_t) = 1\!\!1_{\{t<\tau\}} \frac{P(t < \tau \le s \,|\, \mathcal{F}_t)}{P(t < \tau \,|\, \mathcal{F}_t)} \\ &= 1\!\!1_{\{t<\tau\}} E(1 - \exp(\Lambda_s - \Lambda_t) \,|\, \mathcal{F}_t) \end{aligned}$$

On the other hand, from part (i)

$$\begin{aligned} E\Big(\int_{t\wedge\tau}^{s\wedge\tau} \lambda(X_u) du \,\Big|\, \mathcal{G}_t\Big) &= E\big(\Lambda_{s\wedge\tau} - \Lambda_{t\wedge\tau} \,|\, \mathcal{G}_t\big) \\ &= 1\!\!1_{\{t<\tau\}} E\Big(\int_t^\infty h_u \lambda_u e^{-(\Lambda_u - \Lambda_t)} du \,\Big|\, \mathcal{F}_t\Big) \end{aligned}$$

where $h_u = \Lambda(s \wedge u) - \Lambda(t \wedge u)$. Consequently,

$$\int_t^\infty h_u \lambda_u e^{-(\Lambda_u - \Lambda_t)} du$$

$$= \int_t^s (\Lambda_u - \Lambda_s)\lambda_u e^{-(\Lambda_u - \Lambda_t)} du + (\Lambda_t - \Lambda_s) \int_s^\infty \lambda_u e^{-(\Lambda_u - \Lambda_t)} du$$

$$= -(\Lambda_s - \Lambda_t) e^{-(\Lambda_s - \Lambda_t)} + \int_t^s \lambda(u) e^{-(\Lambda_u - \Lambda_t)} du + (\Lambda_s - \Lambda_t) e^{-(\Lambda_s - \Lambda_t)}$$

$$= 1 - e^{-(\Lambda_s - \Lambda_t)}.$$

This ends the proof. □

### 3.2.4 Conditional Expectation of $\mathcal{F}_\infty$-Measurable Random Variables

**Lemma 3.3** *Let $X$ be an $\mathcal{F}_\infty$-measurable r.v. $X$. Then*

$$\begin{aligned} E(X 1\!\!1_{\{\tau > t\}} \mid \mathcal{F}_t) &= \exp\Big(-\int_0^t \lambda(X_s)\, ds\Big) E(X \mid \mathcal{F}_t), \\ E(X \mid \mathcal{G}_t) &= E(X \mid \mathcal{F}_t). \end{aligned} \tag{12}$$

PROOF: Let $X$ be an $\mathcal{F}_\infty$-measurable r.v. Then,

$$E(X 1\!\!1_{\{\tau > t\}} \mid \mathcal{F}_t) = E(E(X 1\!\!1_{\{\tau > t\}} \mid \mathcal{F}_\infty) \mid \mathcal{F}_t) = P(\tau > t \mid \mathcal{F}_t) E(X \mid \mathcal{F}_\infty).$$

To prove that $E(X \mid \mathcal{G}_t) = E(X \mid \mathcal{F}_t)$, it suffices to check that

$$E(B_t h(\tau \wedge t) X) = E(B_t h(\tau \wedge t) E(X \mid \mathcal{F}_t))$$

for any $B_t \in \mathcal{F}_t$ and any $h = 1\!\!1_{[0,a]}$. For $t \le a$, the equality is obvious. For $t > a$, we have

$$\begin{aligned} E(B_t 1\!\!1_{\{\tau \le a\}} E(X \mid \mathcal{F}_t)) &= E(B_t E(X \mid \mathcal{F}_t) E(1\!\!1_{\{\tau \le a\}} \mid \mathcal{F}_\infty)) \\ &= E(E(B_t X \mid \mathcal{F}_t) E(1\!\!1_{\{\tau \le a\}} \mid \mathcal{F}_t)) \\ &= E(X B_t E(1\!\!1_{\{\tau \le a\}} \mid \mathcal{F}_t)) = E(B_t X 1\!\!1_{\{\tau \le a\}}) \end{aligned}$$

as expected. □

Let us remark that (12) implies that every **F**-square integrable martingale is a **G**-martingale. However, equality (12) does not apply to any $\mathcal{G}$-measurable random variable; in particular $P(\tau \le t \mid \mathcal{G}_t) = 1\!\!1_{\{\tau \le t\}}$ is not equal to $F_t = P(\tau \le t \mid \mathcal{F}_t)$.

### 3.2.5 Defaultable Zero-Coupon Bond

Similar computations to that of the preceding paragraph show that

$$\begin{aligned} E(\mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{G}_t) &= \mathbb{1}_{\{\tau>t\}} \frac{E(\mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{F}_t)}{E(\mathbb{1}_{\{\tau>t\}} \,|\, \mathcal{F}_t)} \\ &= \mathbb{1}_{\{\tau>t\}} E\Big( \exp\big(-\int_t^T \lambda(X_s)\, ds\big) \,\Big|\, \mathcal{F}_t\Big). \end{aligned}$$

Suppose that the price at time $t$ of a default-free bond paying 1 at maturity $t$ is

$$B(t,T) = E\Big( \exp\big(-\int_t^T r(X_s)\, ds\big) \,\Big|\, \mathcal{F}_t\Big).$$

The value of a defaultable zero-coupon bond is

$$\begin{aligned} & E\Big( \mathbb{1}_{\{T<\tau\}} \exp\big(-\int_t^T r(X_s)\, ds\big) \,\Big|\, \mathcal{G}_t\Big) \\ & \quad = \mathbb{1}_{\{\tau>t\}} E\Big( \exp\big(-\int_t^T [r(X_s)+\lambda(X_s)]\, ds\big) \,\Big|\, \mathcal{F}_t\Big). \end{aligned}$$

The $t$-time value of a corporate bond, which pays $\delta$ at time $T$ in case of default and 1 otherwise, is given by

$$E\Big( e^{-\int_t^T r(X_s)\, ds} \big(\delta \mathbb{1}_{\{\tau\le T\}} + \mathbb{1}_{\{\tau>T\}}\big) \,\Big|\, \mathcal{F}_t\Big).$$

The last quantity is equal to

$$\delta B(t,T) + \mathbb{1}_{\{\tau>t\}} (1-\delta) E\Big( \exp\big(-\int_t^T [r(X_s)+\lambda(X_s)]\, ds\big) \,\Big|\, \mathcal{F}_t\Big).$$

It can be proved that, if $h$ is some $\mathbf{F}$-predictable process,

$$E(h_\tau \mathbb{1}_{\{\tau\le T\}} \,|\, \mathcal{G}_t) = h_\tau \mathbb{1}_{\{\tau\le t\}} + \mathbb{1}_{\{\tau>t\}} E\Big( \int_t^T h_u e^{\Lambda_t-\Lambda_u} \lambda(X_u) du \,|\, \mathcal{F}_t\Big).$$

The credit risk model of this kind was studied extensively by Lando [35].

# 4 Hazard Functions of a Random Time

In this section, the problem of quasi-explicit evaluation of various conditional expectations is studied in a very special case when the only filtration available in calculations is the natural filtration of a random time. In practical terms, we consider here an individual who observes the random time, but has no access to any other information. More general situations are examined in the next section.

We start with some well known facts established for the first time in Dellacherie [13] or [14] (p.122), and used again in Chou and Meyer [9], Liptser and Shiryaev [38], Elliott [21], Dellacherie and Meyer [16] (p.237) or more recently in Rogers and Williams [48] and Cocozza-Thivent [10] among others.

Suppose that $\tau$ is an $I\!R_+ \cup \{\infty\}$-valued random variable on some probability space $(\Omega, \mathcal{G}, P)$ such that $P(\tau = 0) = 0$ and $P(\tau > t) > 0, \forall t \in I\!R_+$. As before, we denote by $(N_t\,; t \geq 0)$ the default process, defined as the right-continuous increasing process $N_t = 1\!\!1_{\{\tau \leq t\}}$, and by $\mathcal{H}_t$ its natural filtration $\mathcal{H}_t = \sigma(N_u, u \leq t)$, completed as usual with $P$-negligeable sets. This right-continuous, complete filtration $\mathbf{H}$ is generated by the sets $\{\tau \leq s\}$ for $s \leq t$ (that is the $\sigma$-algebra $\sigma(t \wedge \tau)$) and the atom $\{\tau > t\}$ and is the smallest filtration satisfying the usual hypotheses such that $\tau$ is a $\mathbf{H}$-stopping time.

Notice that any $\mathcal{H}_t$-measurable integrable r.v. $H$ is of the form $H = h(\tau) 1\!\!1_{\{\tau \leq t\}} + \tilde{h} 1\!\!1_{\{\tau > t\}}$ where $h$ is a Borel function defined on $[0, t]$ and $\tilde{h}$ a constant.

**Lemma 4.1** *If $Y$ is any integrable, $\mathcal{G}$-measurable random variable then*

$$E(Y \,|\, \mathcal{H}_t) = 1\!\!1_{\{\tau \leq t\}} E(Y \,|\, \mathcal{H}_\infty) + 1\!\!1_{\{\tau > t\}} \frac{E(Y 1\!\!1_{\{\tau > t\}})}{P(\tau > t)}\,. \tag{13}$$

*In particular,*

$$E(Y \,|\, \mathcal{H}_t) 1\!\!1_{\{\tau > t\}} = 1\!\!1_{\{\tau > t\}} \frac{E(Y 1\!\!1_{\{\tau > t\}})}{P(\tau > t)} \tag{14}$$

*and if $Y$ is $\sigma(\tau)$-measurable, i.e. $Y = h(\tau)$, then*

$$E(Y \,|\, \mathcal{H}_t) \quad = \quad 1\!\!1_{\{\tau \leq t\}} Y + 1\!\!1_{\{\tau > t\}} \frac{E(Y 1\!\!1_{\{\tau > t\}})}{P(\tau > t)}$$

$$= \mathbb{1}_{\{\tau \le t\}} Y + \mathbb{1}_{\{\tau > t\}} \frac{1}{P(\tau > t)} \int_{]t,\infty]} h(u)\,dP(\tau \le u).$$

PROOF: Let $t$ be fixed. The r.v. $E(Y \mid \mathcal{H}_t)$ is $\mathcal{H}_t$-measurable. Therefore, it can be written in the form $E(Y \mid \mathcal{H}_t) = h(\tau)\mathbb{1}_{\{\tau \le t\}} + A\mathbb{1}_{\{\tau > t\}}$ where $A$ is constant and $h$ a Borel function. Multiplying both sides by $\mathbb{1}_{\{\tau > t\}}$, and taking the expectation, we obtain

$$E[\mathbb{1}_{\{\tau > t\}} E(Y \mid \mathcal{H}_t)] = E[E(\mathbb{1}_{\{\tau > t\}} Y \mid \mathcal{H}_t)] = E[\mathbb{1}_{\{\tau > t\}} Y] = AP(\tau > t)\,.$$

It is easy to check that, on the set $\{\tau \le t\}$, we have $E(Y \mid \mathcal{H}_t) = h(\tau) = E(Y \mid \tau) = E(Y \mid \mathcal{H}_\infty)$. In fact, we have given the right-continuous version of the martingale $E(Y \mid \mathcal{H}_t)$. □

## 4.1 Conditional Expectations w.r.t. the Natural Filtration

For easy further reference, let us write down some special cases of the formulae above. For any $t < s$ we have

$$\begin{aligned} P(\tau > s \mid \mathcal{H}_t) &= \mathbb{1}_{\{\tau > t\}} P(\tau > s \mid \tau > t) \\ &= \mathbb{1}_{\{\tau > t\}} \frac{P(\tau > s)}{P(\tau > t)} = \mathbb{1}_{\{\tau > t\}} \frac{1 - F(s)}{1 - F(t)}, \end{aligned} \tag{15}$$

where $F(t) = P(\tau \le t)$ be the right-continuous distribution function of $\tau$. The following result is a straightforward consequence of (15).

**Corollary 4.1** *The process $\widehat{M}$ which equals*

$$\widehat{M}_t = \frac{1 - N_t}{1 - F(t)}, \quad \forall\, t \in I\!R_+, \tag{16}$$

*follows a* $\mathbf{H}$*-martingale. Equivalently, for $t \le s$*

$$E(N_s - N_t \mid \mathcal{H}_t) = (1 - N_t)\frac{F(s) - F(t)}{1 - F(t)} = \mathbb{1}_{\{\tau > t\}} \frac{F(s) - F(t)}{1 - F(t)}. \tag{17}$$

PROOF: The equality (15) can be rewritten as follows

$$E(1 - N_s \mid \mathcal{H}_t) = (1 - N_t)\frac{1 - F(s)}{1 - F(t)}.$$

This immediately yields the martingale property of $\widehat{M}$. □

Before we proceed further, let us recall the standard notion of the *hazard function* of $\tau$.

**Definition 4.1** The increasing function $\Gamma : I\!\!R_+ \to I\!\!R_+$ given by the formula

$$\Gamma(t) = -\ln(1 - F(t)), \quad \forall\, t \in I\!\!R_+, \tag{18}$$

is called the *hazard function* of $\tau$.

It is clear that the relationship $F(t) = e^{-\Gamma(t)}$ is satisfied. If the cumulative distribution function $F$ is an absolutely continuous function, that is, $F(t) = \int_0^t f(u)\,du$, for some function $f : I\!\!R_+ \to I\!\!R_+$, then we have

$$F(t) = 1 - e^{-\Gamma(t)} = 1 - e^{-\int_0^t \gamma(u)\,du}, \quad \forall\, t \in I\!\!R_+,$$

where $\gamma(t) = f(t)(1 - F(t))^{-1}$. It is clear that $\gamma : I\!\!R_+ \to I\!\!R$ is a nonnegative function and satisfies $\int_0^\infty \gamma(u)\,du = \infty$, since $F(\infty) = 1$. The function $\gamma$ is called the *intensity function* (or *hazard rate*) of $\tau$.

Using the hazard function $\Gamma$, we may rewrite (13) as follows

$$E(Y \,|\, \mathcal{H}_t) = 1\!\!1_{\{\tau \le t\}} E(Y \,|\, \tau) + 1\!\!1_{\{\tau > t\}}\, e^{\Gamma(t)}\, E(1\!\!1_{\{\tau > t\}} Y), \tag{19}$$

and (15) takes the form

$$P(\tau > s \,|\, \mathcal{H}_t) = 1\!\!1_{\{\tau > t\}}\, e^{\Gamma(t) - \Gamma(s)}.$$

**Corollary 4.2** *Assume that $Y$ is $\mathcal{H}_\infty$-measurable, so that $Y = h(\tau)$ for some Borel measurable function $h : I\!\!R_+ \to I\!\!R$. If the hazard function $\Gamma$ of $\tau$ is continuous then*

$$E(Y \,|\, \mathcal{H}_t) = 1\!\!1_{\{\tau \le t\}} h(\tau) + 1\!\!1_{\{\tau > t\}} \int_t^\infty h(u) e^{\Gamma(t) - \Gamma(u)}\, d\Gamma(u). \tag{20}$$

*In particular,*

$$E(h(\tau)) = \int_0^\infty h(u) e^{-\Gamma(u)}\, d\Gamma(u)\,.$$

*If $\tau$ admits the intensity function $\gamma$ then*

$$E(Y \,|\, \mathcal{H}_t) = 1\!\!1_{\{\tau \le t\}} h(\tau) + 1\!\!1_{\{\tau > t\}} \int_t^\infty h(u)\gamma(u) e^{-\int_t^u \gamma(v)\,dv}\, du. \tag{21}$$

*In particular, for any $t \leq s$ we have*

$$P(\tau > s \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau > t\}} e^{-\int_t^s \gamma(v)\,dv} \tag{22}$$

*and*

$$P(t < \tau < s \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau > t\}} \Big(1 - e^{-\int_t^s \gamma(v)\,dv}\Big). \tag{23}$$

The following simple result appears to be useful.

**Lemma 4.2** *The process $L$ given by the formula*

$$L_t \stackrel{def}{=} \mathbb{1}_{\{\tau > t\}} e^{\Gamma(t)} = (1 - N_t) e^{\Gamma(t)}, \quad \forall\, t \in I\!\!R_+, \tag{24}$$

*follows a $\mathbf{H}$-martingale.*

Proof: It suffices to observe that $L$ coincides with the process $\widehat{M}$ introduced in Corollary 4.1. □

## 4.2 Applications to the Valuation of Defaultable Claims

Let us fix $T > 0$. We assume that the continuously compounded interest rate $r$ follows a nonnegative deterministic function so that the price at time $t$ of a unit default-free zero-coupon bond of maturity $T$ equals

$$B(t,T) = e^{-\int_t^T r(v)\,dv}, \quad \forall\, t \in [0,T].$$

Our goal is to find quasi-explicit expressions for "values" of certain defaultable claims. Let us assume that $Y = \mathbb{1}_{\{\tau \leq T\}} h(\tau) + \mathbb{1}_{\{\tau > T\}} c$, where $c$ is a constant. If $\Gamma$ is continuous then (20) yields

$$E(Y \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau \leq t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}} \Big( \int_t^T h(u) e^{\Gamma(t) - \Gamma(u)} d\Gamma(u) + c e^{\Gamma(t) - \Gamma(T)} \Big).$$

Similarly, for a fixed $t \leq T$ denote by $Y_t$ the random variable (discounted payoff at time $t$)

$$Y_t = \mathbb{1}_{\{\tau \leq T\}} h(\tau) e^{-\int_t^\tau r(v)\,dv} + \mathbb{1}_{\{\tau > T\}} c e^{-\int_t^T r(v)\,dv}. \tag{25}$$

If $\Gamma$ is an absolutely continuous function, then we get

$$E(Y_t \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau \le t\}} h(\tau) e^{\int_\tau^t r(v)\,dv} + \mathbb{1}_{\{\tau > t\}} \Big( \int_t^T h(u)\gamma(u) e^{-\int_t^u \hat{r}(v)\,dv}\,du + c e^{-\int_t^T \hat{r}(v)\,dv} \Big),$$

where $\hat{r}(v) = r(v) + \gamma(v)$.

(a) The case of a defaultable zero-coupon $T$-maturity bond with zero recovery corresponds to $h = 0$ and $c = 1$ in (25). If we denote the "value" at time $t$ of such a bond by $D^0(t,T)$ then we have

$$D^0(t,T) = \mathbb{1}_{\{\tau > t\}} e^{-\int_t^T (r(v)+\gamma(v))\,dv}, \quad \forall\, t \in [0,T].$$

(b) Assume now that $h = \delta$ for some constant $0 < \delta \le 1$ and $c = 1$. Put more explicitly, we consider the random variable $\widetilde{Y}_t^\delta$ which equals

$$\widetilde{Y}_t^\delta = \mathbb{1}_{\{\tau \le T\}}\, \delta e^{-\int_t^\tau r(v)\,dv} + \mathbb{1}_{\{\tau > T\}}\, e^{-\int_t^T r(v)\,dv}.$$

In this case, for $\widetilde{D}^\delta(t,T) \stackrel{def}{=} E(\widetilde{Y}_t^\delta \,|\, \mathcal{H}_t)$ we get

$$\widetilde{D}^\delta(t,T) = \mathbb{1}_{\{\tau \le t\}} \delta e^{\int_t^\tau r(v)\,dv} + \mathbb{1}_{\{\tau > t\}} \Big( \delta \int_t^T h(u)\gamma(u) e^{-\int_t^u \hat{r}(v)\,dv}\,du + c e^{-\int_t^T \hat{r}(v)\,dv} \Big).$$

Notice that $\widetilde{D}^\delta(t,T)$ represents the value at time $t$ of a $T$-maturity defaultable bond which pays a constant payoff $\delta$ at time of default, if default takes place before maturity date $T$.

(c) Let us finally consider the following random variable

$$Y_t^\delta = \big( \mathbb{1}_{\{\tau \le T\}} \delta + \mathbb{1}_{\{\tau > T\}} \big)\, e^{-\int_t^T r(v)\,dv} = B(t,T) \big( \mathbb{1}_{\{\tau \le T\}} \delta + \mathbb{1}_{\{\tau > T\}} \big).$$

Equivalently, we have

$$Y_t^\delta = \mathbb{1}_{\{\tau \le T\}} \delta e^{-\int_\tau^T r(v)\,dv} e^{-\int_t^\tau r(v)\,dv} + \mathbb{1}_{\{\tau > T\}} e^{-\int_t^T r(v)\,dv},$$

and the last expression leads to $h(\tau) = \delta e^{-\int_\tau^T r(v)\,dv}$, $c = 1$, in formula (25). The above specification of $Y_t^\delta$ corresponds to a defaultable zero-coupon $T$-maturity bond with fractional recovery of par. This means that the bond pays $\delta$ at maturity $T$ if default occurs before maturity (otherwise, it pays the face value 1). For the value $D^\delta(t,T) \stackrel{def}{=} E(Y_t^\delta \,|\, \mathcal{H}_t)$ of such a bond we get

$$\begin{aligned} D^\delta(t,T) &= 1\!\!1_{\{\tau \le t\}} \delta B(t,T) \\ &\quad + 1\!\!1_{\{\tau > t\}} \delta B(t,T)\big(1 - e^{-\int_t^T \gamma(v)\, dv}\big) + 1\!\!1_{\{\tau > t\}} e^{-\int_t^T \hat{r}(v)\, dv}. \end{aligned}$$

## 4.3 Martingale Property of a Continuous Hazard Function

We shall first consider a very special case, when the cumulative distribution function $F$ is an absolutely continuous function, that is, when the random time $\tau$ admits the intensity function $\gamma$. Our goal is to provide the martingale characterization of $\gamma$. To be more specific, we shall check directly that the process

$$M_t = N_t - \int_0^t \gamma(u)\, 1\!\!1_{\{u \le \tau\}}\, du = N_t - \int_0^{t\wedge\tau} \gamma(u)\, du = N_t - \Gamma(t \wedge \tau) \quad (26)$$

follows a $\mathbf{H}$-martingale. To this end, recall that by virtue of (17) we have

$$E(N_s - N_t \,|\, \mathcal{H}_t) = 1\!\!1_{\{\tau > t\}} \frac{F(s) - F(t)}{1 - F(t)}.$$

On the other hand, if we denote

$$Y = \int_t^s \gamma(u) 1\!\!1_{\{u \le \tau\}}\, du = \int_{t\wedge\tau}^{s\wedge\tau} \frac{f(u)}{1 - F(u)}\, du = \ln \frac{1 - F(t \wedge \tau)}{1 - F(s \wedge \tau)}$$

then obviously $Y = 1\!\!1_{\{\tau > t\}} Y$. Using (13), we get

$$\begin{aligned} E(Y \,|\, \mathcal{H}_t) &= 1\!\!1_{\{\tau > t\}} \frac{E(Y)}{P(\tau > t)} = 1\!\!1_{\{\tau > t\}} \frac{\int_t^s \gamma(u)(1 - F(u))\, du}{1 - F(t)} \\ &= 1\!\!1_{\{\tau > t\}} \frac{F(s) - F(t)}{1 - F(t)}. \end{aligned}$$

This shows that the process $M$ defined by (26) follows a $\mathbf{H}$-martingale.

**Lemma 4.3** *Assume that $F(t) = 1 - e^{-\int_0^t \gamma(u)\, du}$ for some function $\gamma : I\!\!R_+ \to I\!\!R_+$. Then the process $M$ given by* (26) *follows a* $\mathbf{H}$*-martingale.*

It appears that Lemma 4.3 remains valid when $F$ is merely continuous. More precisely, we have the following result.

**Proposition 4.1** *Assume that $F$ (and thus also $\Gamma$) is a continuous function. Then the process $M_t = N_t - \Gamma(t \wedge \tau)$ follows a $\mathbf{H}$-martingale.*

PROOF: For sake of brevity, we prefer to make use of Lemma 4.2, rather than to rely on direct calculations. It is clear that $M$ is $\mathbf{H}$-adapted. Using Itô's formula, we obtain (notice that $\Gamma$ can be seen as a continuous process of bounded variation)

$$L_t = (1 - N_t)e^{\Gamma(t)} = 1 + \int_0^t e^{\Gamma(u)}\big((1 - N_u)\, d\Gamma(u) - dN_u\big). \tag{27}$$

This in turn yields

$$M_t = N_t - \Gamma(t \wedge \tau) = \int_0^t \big(dN_u - (1 - N_u)\, d\Gamma(u)\big) = -\int_0^t e^{-\Gamma(u)}\, dL_u,$$

and thus $M$ is a $\mathbf{H}$-martingale. □

In the general case, that is, when $F$ is no longer assumed to be a continuous function, we denote by $F(t-) = P(\tau < t)$ the left-hand side limit of $F$ at $t$.

**Proposition 4.2** *The process $(M_t\,;\, t \geq 0)$ where*

$$M_t \stackrel{def}{=} N_t - \int_{]0, \tau \wedge t]} \frac{dF(s)}{1 - F(s-)} \tag{28}$$

*is a $\mathbf{H}$-martingale.*

PROOF: From Lemma 4.1, for $t > s$

$$\begin{aligned} E(N_t - N_s \,|\, \mathcal{H}_s) &= E(\mathbb{1}_{\{s < \tau \leq t\}} \,|\, \mathcal{H}_s) \\ &= \mathbb{1}_{\{s < \tau\}} A + \mathbb{1}_{\{\tau \leq s\}}\, E(\mathbb{1}_{\{s < \tau \leq t\}} \,|\, \mathcal{H}_\infty) = \mathbb{1}_{\{s < \tau\}} A. \end{aligned}$$

We have proved that the constant $A$ is equal to

$$\frac{P(s < \tau \leq t)}{P(s < \tau)} = \frac{F(t) - F(s)}{1 - F(s)}\,.$$

On the other hand,

$$E\Big(\int_{]\tau\wedge s,\tau\wedge t]}\frac{dF(u)}{1-F(u-)}\,\Big|\,\mathcal{H}_s\Big)=1\!\!1_{\{s<\tau\}}g(s),$$

where

$$g(s)=\frac{1}{1-F(s)}E\Big(1\!\!1_{\{s<\tau\}}\int_{]\tau\wedge s,\tau\wedge t]}\frac{dF(u)}{1-F(u-)}\Big)$$

$$=\frac{1-F(t)}{1-F(s)}\int_{]s,t]}\frac{dF(u)}{1-F(u-)}-\frac{1}{1-F(s)}\int_{]s,t]}dF(v)\int_{]s,v]}\frac{dF(u)}{1-F(u-)}\,.$$

Applying Fubini's theorem, we conclude that $g(s)=\dfrac{F(t)-F(s)}{1-F(s)}$. □

**Proposition 4.3** *Assume that $\Gamma$ is a continuous function. Then for any (bounded) Borel measurable function $h:I\!\!R_+\to I\!\!R$, the process*

$$\widehat{M}^h_t=1\!\!1_{\{\tau\le t\}}h(\tau)-\int_0^{t\wedge\tau}h(u)\,d\Gamma(u) \tag{29}$$

*is a* **H**-*martingale.*

Proof: Notice that the proof given belows provides an alternative proof of Proposition 4.1. We wish to establish through the direct calculations the martingale property of the process $\widehat{M}^h$ given by formula (29).

First, formula (20) in Corollary 4.2 gives

$$I:=E\big(h(\tau)1\!\!1_{\{t<\tau\le s\}}\,|\,\mathcal{H}_t\big)=\;1\!\!1_{\{\tau>t\}}e^{\Gamma(t)}\int_t^s h(u)e^{-\Gamma(u)}\,d\Gamma(u).$$

On the other hand, it is obvious that

$$J:=E\Big(\int_{t\wedge\tau}^{s\wedge\tau}h(u)\,d\Gamma(u)\,\Big|\,\mathcal{H}_t\Big)=E\big(\tilde{h}(\tau)1\!\!1_{\{t<\tau\le s\}}+\tilde{h}(s)1\!\!1_{\{\tau>s\}}\,|\,\mathcal{H}_t\big)$$

where we set $\tilde{h}(s)=\int_t^s h(u)\,d\Gamma(u)$. Consequently, again by (20)

$$J=\;1\!\!1_{\{\tau>t\}}e^{\Gamma(t)}\Big(\int_t^s\tilde{h}(u)e^{-\Gamma(u)}\,d\Gamma(u)+e^{-\Gamma(s)}\tilde{h}(s)\Big).$$

To conclude the proof, it is enough to observe that Fubini's theorem yields

$$\begin{aligned}
&\int_t^s e^{-\Gamma(u)} \int_t^u h(v)\,d\Gamma(v)\,d\Gamma(u) + e^{-\Gamma(s)}\tilde{h}(s) \\
= &\int_t^s h(u) \int_u^s e^{-\Gamma(v)}\,d\Gamma(v)\,d\Gamma(u) + e^{-\Gamma(s)} \int_t^s h(u)\,d\Gamma(u) \\
= &\int_t^s h(u)e^{-\Gamma(u)}\,d\Gamma(u),
\end{aligned}$$

as expected. □

Observe that the last property follows also from Proposition 4.2 combined with the fact that the integral with respect to the martingale $M$ of any predictable process is a martingale.

**Corollary 4.3** *Let $h : I\!R_+ \to I\!R$ be a (bounded) Borel measurable function. Then the process*

$$\widetilde{M}_t^h = \exp\left(1\!\!1_{\{\tau \le t\}} h(\tau)\right) - \int_0^{t\wedge\tau} (e^{h(u)} - 1)\,d\Gamma(u) \tag{30}$$

*is a* **H**-*martingale.*

PROOF: In view of the preceding result applied to $e^h - 1$, it is enough to observe that

$$\exp\left(1\!\!1_{\{\tau\le t\}} h(\tau)\right) = 1\!\!1_{\{\tau\le t\}} e^{h(\tau)} + 1\!\!1_{\{\tau\ge t\}} = 1\!\!1_{\{\tau\le t\}}(e^{h(\tau)} - 1) + 1\,.$$

□

The natural question which arises in this context reads: does the martingale property of the process $M$ introduced above uniquely characterize the intensity function (or, more generally, the hazard function) of $\tau$? To examine this problem, it is useful to notice that the process $A_t = \Gamma(t \wedge \tau)$ satisfies: (i) $A$ is an increasing, right-continuous, **H**-adapted process, and (ii) $N - A$ is a **H**-martingale. It is thus clear that $A$ is a dual predictable projection (or **H**-*compensator*) of the increasing, right-continuous, **H**-adapted process $N$. We shall see that the answer to the question above is positive when the hazard function $\Gamma$ is a continuous function. Otherwise, the answer appears to be negative, that is, the **H**-compensator $A$ of $N$ does not specify the hazard function $\Gamma$ through the relationship $A_t = \Gamma(t \wedge \tau)$, in general. Indeed, when $\Gamma$ has discontinuities then (27) takes the following form

$$L_t \;=\; L_0 + \int_{]0,t]} (1 - N_u)\, de^{\Gamma(u)} - \int_{]0,t]} e^{\Gamma(u-)}\, dN_u,$$

that is (we write $\Delta\Gamma(s) = \Gamma(s) - \Gamma(s-)$),

$$\begin{aligned} L_t \;&=\; 1 + \int_{]0,t]} e^{\Gamma(u-)}\big((1 - N_u)\, d\Gamma(u) - dN_u\big) \\ &+ \sum_{s \le t,\, s < \tau} \big(e^{\Gamma(s)} - e^{\Gamma(s-)} - e^{\Gamma(s-)}\, \Delta\Gamma(s)\big). \end{aligned}$$

Let us stress that both $A$ and $\Gamma$ exist for arbitrary random time $\tau$, and are unique.

## 4.4 Representation Theorem

The following theorem is well known (see Brémaud [7]).

**Theorem 4.1** *Suppose that $F$ is differentiable. Let $H_t = E(h(\tau)\,|\,\mathcal{H}_t)$ for some bounded Borel measurable function $h : I\!R \to I\!R$. Then*

$$H_t = H_0 + \int_{]0,t]} \widetilde{h}(u)\, dM_u, \tag{31}$$

*where $M_t = N_t - \Gamma(t \wedge \tau)$, and the function $\widetilde{h}$ equals*

$$\widetilde{h}(t) = h(t) - e^{\Gamma(t)}\, E\big(h(\tau) 1\!\!1_{\{\tau > t\}}\big).$$

PROOF: Observe first that $H_0 = E(h(\tau))$. Recall also that $H_t$ admits the representation (cf. (20))

$$H_t = E(h(\tau)\,|\,\mathcal{H}_t) = 1\!\!1_{\{\tau \le t\}} h(\tau) + 1\!\!1_{\{\tau > t\}} e^{\Gamma(t)} b(t), \tag{32}$$

where

$$b(t) \stackrel{def}{=} E\big(1\!\!1_{\{\tau > t\}} h(\tau)\big) = \int_t^\infty h(u)\, dF(u) = \int_t^\infty h(u) f(u)\, du.$$

If (31) holds for some function $\widetilde{h}$, then on the set $\{t < \tau\}$ we have

$$H_t = E(h(\tau)) - \int_0^t \widetilde{h}(s)\gamma(s)\,ds = E(h(\tau)) - \int_0^t \widetilde{h}(s)e^{\Gamma(s)}f(s)\,ds,$$

and, in view of (32), $H_t$ also equals $e^{\Gamma_t}b(t)$ on this set. By differentiation of both expressions with respect to $t$, we obtain

$$-\widetilde{h}(t)f(t)e^{\Gamma(t)} = -e^{\Gamma(t)}h(t)f(t) + e^{2\Gamma(t)}f(t)b(t).$$

The equality $\widetilde{h}(t) = h(t) - e^{\Gamma(t)}b(t)$ is thus straightforward on the set $\{t < \tau\}$. Since the process $H$ is manifestly continuous on this set, we also have $\widetilde{h}(t) = h(t) - H_t = h(t) - H_{t-}$ on $\{t < \tau\}$. In view of the last equality, it is clear that on the set $\{\tau \le t\}$ the right-hand side in (31) gives $h(\tau)$, as expected. □

Notice that representation (31) can also be rewritten as follows (cf. formula (58))

$$H_t = H_0 + \int_{]0,t]} (h(u) - H_{u-})\,dM_u. \tag{33}$$

## 4.5 Martingale Characterization of the Hazard Function

We shall now examine the general case, that is, we no longer assume that $\tau$ admits the intensity function $\gamma$, that is, the probability law of $\tau$ is not necessarily absolutely continuous. Let us notice that $N_t = N_{t\wedge\tau}$ (i.e., process $N$ is stopped at time $\tau$) and $E(N_t) = P(\tau \le t) = F(t)$. Consider a function $\Lambda : I\!R_+ \to I\!R$ with $\Lambda(0) = 0$.

**Definition 4.2** A function $\Lambda : I\!R_+ \to I\!R$ is called a *martingale hazard function* of a random time $\tau$ with respect to the natural filtration $\mathbf{H}$ if and only if the process $N_t - \Lambda(t \wedge \tau)$ is a $\mathbf{H}$-martingale.

The function $\Lambda$ can also be seen as a $\mathbf{F}^0$-adapted right-continuous stochastic process, where $\mathbf{F}^0$ is the trivial filtration, $\mathcal{F}_t^0 = \{\emptyset, \Omega\}$ for every $t \in I\!R_+$. We shall sometimes find it useful to refer to the martingale hazard function as the $\mathbf{F}^0$-*martingale hazard process* of a random time $\tau$. The reason for this convention will become clear in Section 5.3, where the notion of a $\mathbf{F}$-martingale hazard process with respect to a non-trivial filtration $\mathbf{F}$ is examined.

**Proposition 4.4** (i) *The (unique) martingale hazard function of $\tau$ with respect to* $\mathbf{H}$ *is the right-continuous increasing function $\Lambda$ given by the formula*

$$\Lambda(t) = \int_{]0,t\,]} \frac{dF(u)}{1 - F(u-)} = \int_{]0,t\,]} \frac{dP(\tau \le u)}{1 - P(\tau < u)}, \quad \forall\, t \in I\!R_+. \tag{34}$$

(ii) *The martingale hazard function $\Lambda$ is continuous if and only if the c.d.f. $F$ is continuous. In this case, $\Lambda$ satisfies $\Lambda(t) = -\ln(1 - F(t))$ (equivalently, $F(t) = 1 - e^{-\Lambda(t)}$).*
(iii) *The martingale hazard function $\Lambda$ coincides with the hazard function $\Gamma$ if and only if $F$ is a continuous function. In general*

$$e^{-\Gamma(t)} = e^{-\Lambda^c(t)} \prod_{0 \le u \le t} (1 - \Delta\Lambda(u)), \tag{35}$$

*where $\Lambda^c(t) = \Lambda(t) - \sum_{0 \le u \le t} \Delta\Lambda(u)$, and $\Delta\Lambda(u) = \Lambda(u) - \Lambda(u-)$.*
(iv) *If $F$ is absolutely continuous then*

$$\Lambda(t) = \Gamma(t) = \int_0^t f(u)(1 - F(u))^{-1}\, du. \tag{36}$$

PROOF: The definition of $\Lambda$ implies that $E(N_t) = E(\Lambda(t \wedge \tau))$, i.e.,

$$F(t) = \int_{]0,t\,]} \Lambda(u)\, dF(u) + \Lambda(t)(1 - F(t)) \tag{37}$$

and thus $\Lambda$ follows a right-continuous function. Moreover, if $\Lambda_1$ and $\Lambda_2$ are right-continuous functions which satisfy (37) then for every $t \in I\!R_+$

$$\int_{]0,t\,]} (\Lambda_1(u) - \Lambda_2(u))\, dF(u) + (\Lambda_1(t) - \Lambda_2(t))(1 - F(t)) = 0.$$

This shows that the martingale hazard function $\Lambda$, if it exists, is unique.

To establish (i), it is enough to check that for any $t \le s$ we have[5] (cf. (17))

$$E(N_s - N_t \,|\, \mathcal{H}_t) = 1\!\!1_{\{\tau > t\}} \frac{F(s) - F(t)}{1 - F(t)} = E(Y \,|\, \mathcal{H}_t),$$

[5] This property was already proved in Proposition 4.2. The proof provided here is based on slightly different arguments, however.

where we have set
$$Y = \int_{]t\wedge\tau, s\wedge\tau]} \frac{dF(u)}{1-F(u-)}.$$
It is clear that $Y = 1\!\!1_{\{\tau>t\}} Y$. Therefore, using (13), we obtain
$$E(Y \mid \mathcal{H}_t) = E(1\!\!1_{\{\tau>t\}} Y \mid \mathcal{H}_t) = 1\!\!1_{\{\tau>t\}} \frac{E(Y)}{1-F(t)}.$$
Furthermore
$$E(Y) = P(\tau > s) \int_{]t,s]} \frac{dF(u)}{1-F(u-)} + \int_{]t,s]} \int_{]t,u]} \frac{dF(v)}{1-F(v-)}\, dF(u)$$
and thus
$$E(Y) = (\Lambda(s) - \Lambda(t))(1-F(s)) + \int_{]t,s]} (\Lambda(u) - \Lambda(t))\, dF(u)$$
$$= (\Lambda(s) - \Lambda(t))(1-F(s)) - \Lambda(t)(F(s) - F(t)) + \int_{]t,s]} \Lambda(u)\, dF(u).$$
The integration by parts formula yields
$$\int_{]t,s]} \Lambda(u)\, dF(u) = \Lambda(s)F(s) - \Lambda(t)F(t) - \int_{]t,s]} F(u-)\, d\Lambda(u).$$
Finally, it is clear from (34) that
$$\int_{]t,s]} F(u-)\, d\Lambda(u) = \Lambda(s) - \Lambda(t) - F(s) + F(t).$$
Combining the equalities above, we find that $E(Y) = F(s) - F(t)$ for every $t \leq s$. This completes the proof of (i). Statements (ii)-(iv) are almost immediate consequences of (i) and the definition of a hazard function. Let us only observe that at any point of discontinuity of $F$ we have
$$\Delta\Lambda(t) \stackrel{def}{=} \Lambda(t) - \Lambda(t-) = \frac{F(t) - F(t-)}{1-F(t-)}.$$
On the other hand, for the hazard function $\Gamma$ we obtain
$$e^{-\Delta\Gamma(t)} = e^{-(\Gamma(t)-\Gamma(t-))} = \frac{1-F(t)}{1-F(t-)} = 1 - \Delta\Lambda(t). \tag{38}$$

This shows that the martingale hazard function $\Lambda$ and the hazard function $\Gamma$ cannot coincide when $F$ is discontinuous. In view of (38), relationship (35) is also easy to establish. As was already mentioned, the notion of a martingale hazard function is closely related to the **H**-compensator of $\tau$ (or rather, the **H**-compensator of the associated jump process $N$). Let us first recall the definition of a compensator of an increasing process. In our context, it can be stated as follows.

**Definition 4.3** A process $A$ is called a **H**-*compensator* of the jump process $N$ if and only if the following hold:
(i) $A$ is an **H**-predictable right-continuous increasing process, $A_0 = 0$,
(ii) the process $N - A$ is a **H**-martingale.

Using the well-known result on the existence and uniqueness of the Doob-Meyer decomposition with respect to the filtration **H** which satisfies the 'usual conditions,' it is easy to check that a process $A$ is a **H**-compensator of the jump process $N$ if and only if $A_t = \Lambda(t \wedge \tau)$, where $\Lambda$ is the martingale hazard function of $\tau$. Therefore, we have the following result.

**Lemma 4.4** *The unique* **H**-*compensator $A$ of a random time $\tau$ is given by the formula*

$$A_t = \int_{]0,t\wedge\tau]} \frac{dF(u)}{1 - F(u-)} = \Lambda(t \wedge \tau), \quad \forall\, t \in I\!\!R_+. \tag{39}$$

PROOF: In view of the definition of the martingale hazard function and Proposition 4.4, it is enough to check that the process $A_t = \Lambda(t \wedge \tau)$, is **H**-predictable. But this is obvious, since $t \to t \wedge \tau$ is a continuous **H**-adapted process, so that it is **H**-predictable. □

Combining part (ii) in Proposition 4.4 with Lemma 4.4 we get immediately the following corollary.

**Corollary 4.4** *The hazard function $\Gamma$ of $\tau$ is related to the* **H**-*compensator $A$ of the jump process $N$ through the formula $A_t = \Gamma(t \wedge \tau)$ if and only if the cumulative distribution function $F$ of $\tau$ is continuous.*

## 4.6 Duffie and Lando's Result

We shall examine a model studied in Duffie and Lando [19]. They assume that $\tau = \tau_0 = \inf\{t \geq 0 : V_t = 0\}$, where the process $V$ satisfies

$$dV_t = \mu(t, V_t)\,dt + \sigma(t, V_t)\,dW_t, \quad V_0 = v > 0, \tag{40}$$

where $W$ is a Brownian motion. Suppose there is a risky asset and a riskless one with zero rate, such that there exists a unique equivalent martingale measure on $\mathcal{F}_T$. Then, any $\mathcal{F}_T$-measurable square-integrable r.v. is the terminal value of a self financing strategy, and we shall say that the market is $\mathcal{F}_T$-complete. The price of the defaultable zero-coupon bond would be $E_Q(1\!\!1_{\{T<\tau_0\}})$ in this $\mathcal{F}_T$-complete market, $Q$ being the risk neutral probability, and the hedging strategy would be similar to the case of barrier option. The time $\tau_0$ is a stopping time with respect to the Brownian filtration $\mathcal{F}_t = \sigma(W_s,\, s \leq t)$. Therefore, it is predictable in that filtration and admits no intensity. We shall discuss this point later.

Here, we suppose, as in [19], that the agent will observe default when it happens but will have no knowledge of $V$ before default has occured. In this case, when the default has not yet appeared, the value of a zero-coupon is given in terms of the hazard function of $N$ as $\exp\big(\Gamma(T) - \Gamma(t)\big)$, where $d\Gamma(s) = \dfrac{dF(s)}{1 - F(s)}$ and $F(s) = P(\tau \leq s)$ (assumed to be continuous). The next result is a general fact and remains true for any default time, without any additional hypothesis.

**Proposition 4.5** *Let $V$ be a diffusion whose dynamics are given by* (40) *and $\tau$ a stopping time which models the default time. The hazard function of $\tau$ in the filtration $\mathbf{H}$ is $\Gamma(t) = -\ln(1 - F(t))$, and the value of a defaultable zero-coupon bond is*

$$E\Big(e^{-\int_t^T r(s)\,ds}\, 1\!\!1_{\{T<\tau\}} \,\Big|\, \mathcal{H}_t\Big) = 1\!\!1_{\{\tau>t\}}\, e^{-\int_t^T \big(r(s)\,ds + d\Gamma(s)\big)},$$

*where $r$ is the deterministic interest rate.*

Duffie and Lando [19] have shown that the intensity function of $\tau_0$ equals

$$\lambda(t) = \frac{1}{2}\sigma^2(t, 0)\frac{\partial \varphi}{\partial x}(t, 0),$$

where $\varphi(t,x)$ is the conditional density of $V_t$ when $t < \tau_0$, i.e., the derivative with respect to $x$ of $\dfrac{P(V_t \le x, t < \tau_0)}{P(t < \tau_0)}$. The equivalence between Duffie-Lando's and our result is obvious. In fact, Duffie and Lando represent $\lambda(t)$ in the following way

$$\lambda(t) = \lim_{h \to 0} \frac{1}{hP(t < \tau_0)} \int_0^\infty P(V_t \in dx, t < \tau_0) P_x(\tau_0 < h) \tag{41}$$

and they establish that this limit is equal to $\dfrac{1}{2}\sigma^2(t,0)\dfrac{\partial \varphi}{\partial x}(t,0)$. The right-hand side of (41) can be written as

$$\begin{aligned}
& \frac{1}{hP(t < \tau_0)} \int_0^\infty P(V_t \in dx, t < \tau_0)(1 - P_x(h < \tau_0)) \\
= \quad & \frac{1}{hP(t < \tau_0)} \Big( P(t < \tau_0) - \int_0^\infty P(V_t \in dx, t + h < \tau_0) \Big) \\
= \quad & \frac{1}{hP(t < \tau_0)} (P(t < \tau_0) - P(t + h < \tau_0)) = \frac{f(t)}{1 - F(t)} \, ,
\end{aligned}$$

which is our result. The proof that the limit in (41) is $\dfrac{1}{2}\sigma^2(t,0)\dfrac{\partial \varphi}{\partial x}(t,0)$ is quite complicated, however. Duffie and Lando prove first this result for a Brownian motion, then for the Ornstein-Uhlenbeck process, and finally for more general diffusion processes. See also [22] for another proof.

## 4.7 Generalizations

We give here some ideas how to extend the previous model.

### 4.7.1 Successive default times

First, the previous results can easily be generalized to the case of successive default times. We reproduce here the result of [9]. Let $\tau_k$ be successive times of default, $N_t = \sum_k \mathbb{1}_{\{\tau_k \le t\}}$ and $\mathbf{H}$ the canonical filtration of $N$. Let us introduce $T_k = \tau_k - \tau_{k-1}$. Then, the process $N_t - A_t$ is an $\mathbf{H}$-martingale, where

$$A_t = \phi_1(T_1) + \ldots + \phi_{n-1}(T_1, \ldots, T_{n-2}; T_{n-1}) + \phi_n(T_1, \ldots, T_{n-1}; t - T_n)$$

on $\tau_n \leq t < \tau_{n+1}$, and

$$\phi_k(t_1, \ldots, t_{k-1}; t) = \int_{]0,t]} \frac{dF_k(s_1, \ldots, s_{k-1}; s)}{1 - F_k(s_1, \ldots, s_{k-1}; s-)}$$

$$F_k(t_1, \ldots, t_{k-1}; t) = P(\tau_k \leq t \,|\, T_1 = t_1, \ldots, T_{n-1} = t_{n-1}).$$

### 4.7.2 Partial observations

Let us assume that the agent observes $\mathcal{H}_t$ as well as the prices (or the value of the firm) at some discrete times, say for each time $t_1 < t_2 < \ldots < t_k$. The information of the agent is $\mathcal{G}_t = \mathcal{H}_t \vee \sigma(S_{t_k}, t_k < t)$. Let $G$ be a $\mathbf{G}$-adapted process. Then,

$$G_t 1\!\!1_{\{t<\tau\}} = 1\!\!1_{\{t<\tau\}} \sum_k g_{k-1} 1\!\!1_{\{t_{k-1} \leq \tau < t_k\}},$$

where $g_k$ is $\sigma(S_{t_j}, j \leq k)$-measurable. The $\mathbf{G}$-intensity of default is defined as the $\mathbf{G}$-adapted process $\Lambda$ such that $N_t - \int_0^{t\wedge\tau} d\Lambda_s$ is a martingale. The same method as in the previous section leads to $d\Lambda_s = \dfrac{dF(s)}{1 - F(s-)}$, where $F(s) = P(\tau \leq s \,|\, \sigma(S_{t_j}, t_j \leq s))$.

# 5 Hazard Processes of a Random Time

In this section, previously introduced concepts are extended to the case when a larger flow of information – formally represented by a filtration $\mathbf{F}$ – is available. Generally speaking, our goal is to examine formulae for the conditional expectation of the form $E(Y \,|\, \mathcal{G}_t)$, where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ (at least for certain classes of $\mathcal{G}$-measurable random variables $Y$).

## 5.1 Conditional Expectations w.r.t. an Arbitrary Filtration

As before, we denote by $\tau$ a nonnegative random variable on a probability space $(\Omega, \mathcal{G}, P)$, such that $P(\tau = 0) = 0$ and $P(\tau > t) > 0$ for any $t \in I\!R_+$. As usual, we introduce a right-continuous process $N$ by setting

$N_t = 1\!\!1_{\{\tau \le t\}}$, and we denote by $\mathbf{H}$ the associated filtration: $\mathcal{H}_t = \sigma(N_u : u \le t)$.

We assume that we are given a natural filtration of a certain Brownian motion $\mathbf{F}$ and we define $\mathbf{G} = \mathbf{H} \vee \mathbf{F}$, that is, $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for every $t$.[6] For each $t$, the $\sigma$-field $\mathcal{G}_t$ is assumed to represent all observations available at time $t$.

The process $N$ is obviously $\mathbf{G}$-adapted, but not necessarily $\mathbf{F}$-adapted. In other words, $\tau$ is a $\mathbf{G}$-stopping time, but not necessarily a $\mathbf{F}$-stopping time (see [50] for a more general case).

**Proposition 5.1** *$\mathcal{F}_t \subset \mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t \subset \mathcal{G}^*_t$, for every $t \in I\!\!R_+$, where*

$$\mathcal{G}^*_t \stackrel{def}{=} \big\{ A \in \mathcal{G} \,|\, \exists B \in \mathcal{F}_t \ \ A \cap \{\tau > t\} = B \cap \{\tau > t\} \big\}.$$

*For any $t \in I\!\!R_+$ and for any event $A \in \mathcal{H}_\infty \vee \mathcal{F}_t$ we have $A \cap \{\tau \le t\} \in \mathcal{G}_t$.*

PROOF: Observe that $\mathcal{G}_t \subset \mathcal{H}_t \vee \mathcal{F}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t) = \sigma(\{\tau \le u\}, u \le t, \mathcal{F}_t)$. Also, it is easily seen that the class $\mathcal{G}^*_t$ is a sub-$\sigma$-field of $\mathcal{G}$. Therefore, it is enough to check that if either $A = \{\tau \le u\}$ for some $u \le t$ or $A \in \mathcal{F}_t$, then there exists an event $B \in \mathcal{F}_t$ such that $A \cap \{\tau > t\} = B \cap \{\tau > t\}$. Indeed, in the former case we may take $B = \emptyset$, in the latter $B = A$. □

### 5.1.1 Hazard Process $\Gamma$

For any $t \in I\!\!R_+$, we write $F_t = P(\tau \le t \,|\, \mathcal{F}_t)$, so that $1 - F_t = P(\tau > t \,|\, \mathcal{F}_t)$. It is easily seen that the process $F$ ($1 - F$, resp.) is a bounded, nonnegative $\mathbf{F}$-submartingale ($\mathbf{F}$-supermartingale, resp.) We may thus deal with the right-continuous modification of $F$. The next definition is a straightforward generalization of Definition 4.1.

**Definition 5.1** Assume that $F_t < 1$ for every $t \in I\!\!R_+$. The $\mathbf{F}$-*hazard process* of $\tau$, denoted by $\Gamma$, is defined through the formula $1 - F_t = e^{-\Gamma_t}$, or equivalently, $\Gamma_t = -\ln(1 - F_t)$ for every $t \in I\!\!R_+$.

[6] Recall that all filtrations are assumed to be $(P, \mathcal{G})$-completed. We assume also that the enlarged filtration $\mathbf{G}$ satisfies the 'usual conditions.'

In Section 5.1, we assume throughout that the inequality $F_t < 1$ holds for every $t$, so that the $\mathbf{F}$-hazard process $\Gamma$ is well defined. It should be stressed that the case when $\tau$ is a $\mathbf{F}$-stopping time, is not examined in this section.

### 5.1.2 Conditional Expectation $E(\mathbb{1}_{\{\tau>t\}}Y \mid \mathcal{G}_t)$

We start with the following result, which is a generalization of Lemma 4.1.

**Lemma 5.1** *For any $\mathcal{G}$-measurable bounded random variable $Y$ we have, for any $t \in I\!R_+$,*

$$\begin{aligned} E(\mathbb{1}_{\{\tau>t\}}Y \mid \mathcal{G}_t) &= \mathbb{1}_{\{\tau>t\}} \frac{E(\mathbb{1}_{\{\tau>t\}}Y \mid \mathcal{F}_t)}{P(\tau > t \mid \mathcal{F}_t)} \\ &= \mathbb{1}_{\{\tau>t\}} E(\mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} Y \mid \mathcal{F}_t), \end{aligned} \tag{42}$$

*for any $t \le s$*

$$\begin{aligned} P(t < \tau \le s \mid \mathcal{G}_t) &= \mathbb{1}_{\{\tau>t\}} \frac{P(t < \tau \le s \mid \mathcal{F}_t)}{P(\tau > t \mid \mathcal{F}_t)} \\ &= \mathbb{1}_{\{\tau>t\}} E\big(1 - e^{\Gamma_t - \Gamma_s} \mid \mathcal{F}_t\big). \end{aligned} \tag{43}$$

$$E(\mathbb{1}_{\{\tau>s\}} Y \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} E\big(Y \mathbb{1}_{\{\tau>s\}} e^{\Gamma_t} \mid \mathcal{F}_t\big). \tag{44}$$

*and for any $\mathcal{F}_s$-measurable random variable $X$ we have*

$$E(\mathbb{1}_{\{\tau>s\}} X \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} E\big(X e^{\Gamma_t - \Gamma_s} \mid \mathcal{F}_t\big). \tag{45}$$

PROOF: The proof follows from the remark that the restriction to the set $\{\tau > t\}$ of any $\mathcal{G}_t$-measurable random variable represents a $\mathcal{F}_t$-measurable random variable. Formula (45) follows from

$$E(\mathbb{1}_{\{\tau>s\}} e^{\Gamma_t} X \mid \mathcal{F}_t) = E(e^{\Gamma_t} X\, E(\mathbb{1}_{\{\tau>s\}} \mid \mathcal{F}_s) \mid \mathcal{F}_t). \qquad \square$$

### 5.1.3 Applications to Defaultable Bonds

The following formula, which is a simple consequence of (44), will prove useful is some applications. Let $\delta$ be a constant. Then for arbitrary $\mathcal{G}$-measurable random variable $Y$ we have,

$$E(\mathbb{1}_{\{\tau\le s\}} \delta + \mathbb{1}_{\{\tau>s\}} Y \mid \mathcal{G}_t) = \delta\, P(\tau \le s \mid \mathcal{G}_t) + \mathbb{1}_{\{\tau>t\}} E(\mathbb{1}_{\{\tau>s\}} Y \mid \mathcal{G}_t),$$

so that

$$
\begin{aligned}
&E(1\!\!1_{\{\tau\le s\}}\,\delta + 1\!\!1_{\{\tau>s\}}\,Y \,|\, \mathcal{G}_t) \\
&\quad = \delta\big(1 - 1\!\!1_{\{\tau>t\}}\,E\big(1 - e^{\Gamma_t-\Gamma_s} \,|\, \mathcal{F}_t\big)\big) + 1\!\!1_{\{\tau>t\}}\,E(1\!\!1_{\{\tau>s\}}\,e^{\Gamma_t} Y \,|\, \mathcal{F}_t).
\end{aligned}
$$

If $Y$ is $\mathcal{F}_s$-measurable then we may rewrite the last formula as follows

$$
\begin{aligned}
&E(1\!\!1_{\{\tau\le s\}}\,\delta + 1\!\!1_{\{\tau>s\}}\,Y \,|\, \mathcal{G}_t) \\
&\quad = \delta 1\!\!1_{\{\tau\le t\}} + \delta\,1\!\!1_{\{\tau>t\}}\,E\big(1 - e^{\Gamma_t-\Gamma_s} \,|\, \mathcal{F}_t\big) + 1\!\!1_{\{\tau>t\}}\,E(e^{\Gamma_t-\Gamma_s}\,Y \,|\, \mathcal{F}_t).
\end{aligned}
$$

### 5.1.4 Conditional Expectation $E(Y \,|\, \mathcal{G}_t)$

It is easily seen that

$$
1\!\!1_{\{\tau\le t\}} E(Y \,|\, \mathcal{G}_t) = E(1\!\!1_{\{\tau\le t\}} Y \,|\, \mathcal{H}_\infty \vee \mathcal{F}_t). \tag{46}
$$

First, $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t \subset \mathcal{H}_\infty \vee \mathcal{F}_t$. Furthermore, for any $A \in \mathcal{H}_\infty \vee \mathcal{F}_t$ we have

$$
\begin{aligned}
\int_A E(1\!\!1_D Y \,|\, \mathcal{H}_\infty \vee \mathcal{F}_t)\,dP &= \int_A 1\!\!1_D Y\,dP \\
&= \int_{A\cap D} Y\,dP = \int_{A\cap D} E(Y \,|\, \mathcal{G}_t)\,dP \\
&= \int_A 1\!\!1_D E(Y \,|\, \mathcal{G}_t)\,dP,
\end{aligned}
$$

where we write $D = \{\tau \le t\}$. Notice that the random variable $\mathrm{I}_D E(Y|\mathcal{G}_t)$ is manifestly $\mathcal{G}_t$-measurable, and thus it is also $\mathcal{H}_\infty \vee \mathcal{F}_t$-measurable. We conclude that (46) holds.

By combining (46) with (42), we obtain the following result (notice that formula (47) is a straightforward generalization of equality (13)).

**Lemma 5.2** *For any $\mathcal{G}$-measurable random variable $Y$ we have*

$$
E(Y \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau\le t\}}\,E(Y \,|\, \mathcal{H}_\infty \vee \mathcal{F}_t) + 1\!\!1_{\{\tau>t\}}\,E(\,1\!\!1_{\{\tau>t\}}\,e^{\Gamma_t} Y \,|\, \mathcal{F}_t). \tag{47}
$$

The following corollary is useful in calculations related to defaultable claims.

**Corollary 5.1** *Assume that $F$ (and thus also $\Gamma$) follows a continuous process of bounded variation. Let $Y = h(\tau)$ for a (bounded) Borel measurable function $h : I\!R_+ \to I\!R$. Then*

$$E(Y \mid \mathcal{G}_t) = 1\!\!1_{\{\tau \le t\}}\, h(\tau) + 1\!\!1_{\{\tau > t\}}\, E\Big( \int_t^\infty h(u) e^{\Gamma_t - \Gamma_u}\, d\Gamma_u \,\Big|\, \mathcal{F}_t \Big). \tag{48}$$

*Let $Z$ be a* $\mathbf{F}$*-predictable process. Then for any $t \le s$*

$$E(Z_\tau \mathrm{I}_{\{t < \tau \le s\}} \mid \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}}\, E\Big( \int_t^s Z_u e^{\Gamma_t - \Gamma_u}\, d\Gamma_u \,\Big|\, \mathcal{F}_t \Big). \tag{49}$$

PROOF: Equality (48) follows from (47). Indeed, in view of (47), it is enough to check that the following equality holds

$$\begin{aligned} E( 1\!\!1_{\{\tau > t\}}\, h(\tau) \mid \mathcal{F}_t) &= E\Big( \int_t^\infty h(u) e^{-\Gamma_u}\, d\Gamma_u \,\Big|\, \mathcal{F}_t \Big) \\ &= E\Big( \int_t^\infty h(u)\, dF_u \,\Big|\, \mathcal{F}_t \Big), \end{aligned}$$

where the second equality is a consequence of the equality $dF_u = e^{-\Gamma_u}\, d\Gamma_u$ which holds when $\Gamma$ is a continuous process of bounded variation. We consider first a stepwise function $h(u) = \sum_{i=0}^n h_i 1\!\!1_{[t_i, t_{i+1}]}(u)$, where, w.l.o.g., $t_0 = t < \cdots < t_{n+1} = \infty$.

Then we have (we use here, in particular, formula (45))

$$\begin{aligned} E( 1\!\!1_{\{\tau > t\}} h(\tau) \mid \mathcal{F}_t) &= \sum_{i=0}^n E\big( E( 1\!\!1_{\{\tau > t\}} h_i 1\!\!1_{[t_i, t_{i+1}]}(\tau) \mid \mathcal{F}_{t_{i+1}}) \mid \mathcal{F}_t \big) \\ &= E\Big( \sum_{i=0}^n h_i (F_{t_{i+1}} - F_{t_i}) \,\Big|\, \mathcal{F}_t \Big) \\ &= E\Big( \sum_{i=0}^n \int_{t_i}^{t_{i+1}} h(u)\, dF_u \,\Big|\, \mathcal{F}_t \Big) \\ &= E\Big( \int_t^\infty h(u)\, dF_u \,\Big|\, \mathcal{F}_t \Big). \end{aligned}$$

The proof of (49) is similar. We start by assuming that $Z$ is a stepwise process, so that (we are interested only in $Z_u$ for $u > t$)

$$Z_u = \sum_{i=0}^n Z_{t_i} 1\!\!1_{[t_i, t_{i+1}]}(u),$$

where $t_0 = t < \cdots < t_{n+1} = \infty$. Using Lemma 5.1, we obtain

$$E\big(\mathrm{I}_{\{t_i \le \tau < t_{i+1}\}} Z_\tau \,|\, \mathcal{G}_t\big) = 1\!\!1_{\{\tau > t\}} e^{\Gamma_t} E\big(\mathrm{I}_{\{t_i \le \tau < t_{i+1}\}} Z_{t_i} \,|\, \mathcal{F}_t\big).$$

Then we proceed along the similar lines as in the first part of the proof. $\square$

### 5.1.5 Martingales Associated with the Hazard Process $\Gamma$

**Lemma 5.3** *The process $L_t = 1\!\!1_{\{\tau > t\}} e^{\Gamma_t} = (1 - N_t) e^{\Gamma_t}$ follows a* **G***-martingale.*

PROOF: It is enough to check that for any $t \le s$

$$E(1\!\!1_{\{\tau > s\}} e^{\Gamma_s} \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} e^{\Gamma_t}.$$

In view of (44) this can be rewritten as follows

$$1\!\!1_{\{\tau > t\}} e^{\Gamma_t} E(1\!\!1_{\{\tau > s\}} e^{\Gamma_s} \,|\, \mathcal{F}_t) = 1\!\!1_{\{\tau > t\}} e^{\Gamma_t}.$$

To complete the proof, it is enough to observe that

$$E(1\!\!1_{\{\tau > s\}} e^{\Gamma_s} \,|\, \mathcal{F}_t) = E\big(e^{\Gamma_s} E(1\!\!1_{\{\tau > s\}} \,|\, \mathcal{F}_s) \,|\, \mathcal{F}_t\big) = 1. \qquad \square$$

In the next result, we deal with the continuous case, that is, we assume that $\Gamma$ is a continuous process. The following result is a counterpart of Propositions 4.1-4.3.

**Proposition 5.2** *Assume that the* **F***-hazard process $\Gamma$ of a random time $\tau$ follows a continuous process of bounded variation. Then the process $\hat{M}_t = N_t - \Gamma_{t \wedge \tau}$ follows a* **G***-martingale. Furthermore, for any (bounded)* **F***-predictable process $Z$ the processes*

$$\hat{M}^Z_t = Z_\tau 1\!\!1_{\{\tau \le t\}} - \int_0^{t \wedge \tau} Z_u \, d\Gamma_u \tag{50}$$

*and*

$$\tilde{M}^Z_t = \exp\big(1\!\!1_{\{\tau \le t\}} Z_\tau\big) - \int_0^{t \wedge \tau} (e^{Z_u} - 1) \, d\Gamma_u \tag{51}$$

*are* **G***-martingales*

PROOF: The martingale property of $\hat{M}$ can be shown using the same arguments as in the proof of Proposition 4.1, that is, Itô's lemma combined with Lemma 5.3. To show the martingale property of $\hat{M}^Z$, we proceed along the similar lines as in the proof of Proposition 4.3 (making use of formula (48) in Corollary 5.1, rather than of Corollary 4.2). Finally, it is clear that (51) is an easy consequence of (50). □

**Remark 5.1** If the continuous process $\Gamma$ is not of bounded variation, formula (27) becomes

$$L_t = (1 - N_t)e^{\Gamma_t} = 1 + \int_0^t e^{\Gamma_u}\big((1 - N_u)\,(d\Gamma_u + (1/2)\langle\Gamma\rangle_u) - dN_u\big)$$

and it is no longer true that $\hat{M}$ is a $\mathbf{G}$-martingale.

### 5.1.6 F-Intensity of a Random Time

Let us consider the classic case of an absolutely continuous $\mathbf{F}$-hazard process $\Gamma$. We assume that $\Gamma_t = \int_0^t \gamma_u\, du$ for some $\mathbf{F}$-progressively measurable process $\gamma$, referred to as the $\mathbf{F}$-*intensity* of a random time $\tau$. By virtue of Proposition 5.2, the process $M$ given by the formula

$$M_t = N_t - \int_0^{t\wedge\tau} \gamma_u\, du = N_t - \int_0^t \mathbb{1}_{\{\tau \geq u\}}\gamma_u\, du \tag{52}$$

follows a $\mathbf{G}$-martingale. The property is frequently used in the financial literature as a definition of a $\mathbf{F}$-intensity of a random time. The intuitive meaning of the $\mathbf{F}$-intensity $\gamma$ as the "intensity of survival given the information flow $\mathbf{F}$" becomes clear from the following corollary. However, in this general setting, $\gamma$ is not necessarily a positive process.

**Corollary 5.2** *If the* $\mathbf{F}$*-hazard process* $\Gamma$ *of* $\tau$ *is absolutely continuous then for any* $t \leq s$

$$P(\tau > s\,|\,\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, E\big(e^{-\int_t^s \gamma_u\, du}\,\big|\,\mathcal{F}_t\big)$$

*and*

$$P(t < \tau \leq s\,|\,\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, E\big(1 - e^{-\int_t^s \gamma_u\, du}\,\big|\,\mathcal{F}_t\big).$$

**Remark 5.2** Since obviously the $\mathbf{F}$-hazard function $\Gamma$ is not well defined when $\tau$ is a $\mathbf{F}$-stopping time (that is, when $\mathbf{H} \subset \mathbf{F}$ so that $\mathbf{G} = \mathbf{F}$), Corollary 5.2 cannot be directly applied in such a case. However, it appears that for a certain class of a $\mathbf{G}$-stopping times we can find an increasing $\mathbf{G}$-predictable process $\Lambda$ such that for any $t \leq s$

$$P(\tau > s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} \, E\big(e^{\Lambda_t - \Lambda_s} \,\big|\, \mathcal{G}_t\big) = 1\!\!1_{\{\tau > t\}} \, E\big(e^{-\int_t^s \lambda_u \, du} \,\big|\, \mathcal{G}_t\big),$$

where the second equality holds provided that the process $\Lambda$ is absolutely continuous. It seems natural to conjecture that the process $\Lambda$ (formally introduced in Section 5.3) always represents the $\hat{\mathbf{F}}$-hazard process of $\tau$ for some filtration $\hat{\mathbf{F}}$ such that $\tau$ is not a $\hat{\mathbf{F}}$-stopping time. From this viewpoint, the notion of a martingale hazard process could be seen as a purely technical tool which allows us to find the 'natural' hazard process of $\tau$.

## 5.2 Hypothesis (H) and Extensions

### 5.2.1 Hypothesis (H)

We shall now examine the hypothesis (H) which reads:

**(H)** Every $\mathbf{F}$ square-integrable martingale is a $\mathbf{G}$ square-integrable martingale.

This hypothesis implies that the $\mathbf{F}$-Brownian motion remains a Brownian motion in the enlarged filtration. It was studied by Brémaud and Yor [6] and Mazziotto and Szpirglas [42], and for financial purpose by Kusuoka [33]. This hypothesis is quite natural, despite its technical form. It is equivalent to:

**(H*)** For any $t$, the $\sigma$-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$.

This can be written in any of the equivalent forms (see, e.g. Dellacherie and Meyer [16]) :

**(H1)** $\quad \forall\, F \in \mathcal{F}_\infty,\ \forall\, G_t \in \mathcal{G}_t, \quad E(FG_t \,|\, \mathcal{F}_t) = E(F \,|\, \mathcal{F}_t)\, E(G_t \,|\, \mathcal{F}_t)$

**(H2)** $\quad \forall\, t \geq 0,\ \forall\, G_t \in \mathcal{G}_t, \quad E(G_t \,|\, \mathcal{F}_\infty) = E(G_t \,|\, \mathcal{F}_t)$

**(H3)** $\quad \forall\, t \geq 0,\ \forall\, F \in \mathcal{F}_\infty, \quad E(F \,|\, \mathcal{G}_t) = E(F \,|\, \mathcal{F}_t).$

**Lemma 5.4** *In our setting,* (H) *is equivalent to the following condition* (H$'$)

$$\forall\, s \le t, \quad P(\tau \le s \,|\, \mathcal{F}_\infty) = P(\tau \le s \,|\, \mathcal{F}_t). \tag{53}$$

PROOF: If (H2) holds, then (53) holds too. If (53) holds, the fact that $\mathcal{H}_t$ is generated by the sets $\{\tau \le s\}, s \le t$ proves that $\mathcal{F}_\infty$ and $\mathcal{H}_t$ are conditionally independent given $\mathcal{F}_t$. The property follows. This result can be also found in [15]. □

**Remark 5.3** (i) Equality (53) appears in several papers on default risk, typically without any reference to the (H) hypothesis. For example, in the Madan-Unal paper [41], the main theorem follows from the fact that (53) holds (See the proof of B9 in the appendix of their paper). This is also the case for Wong's model [57].
(ii) If $\tau$ is $\mathcal{F}_\infty$-measurable, and if (53) holds, then $\tau$ is an $\mathbf{F}$-stopping time (and does not admit intensity). If $\tau$ is a $\mathbf{F}$-stopping time, equality (53) holds. Though condition (H) does not necessarily hold true, in general, it is satisfied when $\tau$ is constructed through a standard approach (see Section 7 below).

### 5.2.2 Representation Theorem

For the case of the filtration $\mathbf{F}$ is generated by a Brownian motion Kusuoka [33] establishes the following representation theorem.

**Theorem 5.1** *Any* $\mathbf{G}$*-square integrable martingale admits a representation as a sum of a stochastic integral with respect to the Brownian motion and a stochastic integral with respect to the discontinuous martingale* $M$*.*

PROOF: It suffices to prove that any r.v. of the form $X = (1 - N_s)F_t$, where $s < t$ and $F_t \in \mathcal{F}_t$, can be represented as the sum of two stochastic integrals. From the equality $X = L_t e^{-\Gamma_t} F_t$, and the representation theorem in the filtration $\mathbf{F}$, we obtain $X = (1 + \int_0^s \Gamma_u L_u dM_u)(E(F_t) + \int_0^t \psi_u dB_u)$. □

In this case, the defaultable market is complete as soon as there are a riskless asset and two tradable risky assets, for instance: an asset modelled through a geometric Brownian motion and a defaultable zero-coupon bond.

### 5.3 Martingale Hazard Process $\Lambda$

In view of Proposition 5.2, it is natural to adopt the following definition of the $\mathbf{F}$-martingale hazard process of a random time.

**Definition 5.2** A $\mathbf{F}$-predictable right-continuous increasing process $\Lambda$ is called a $\mathbf{F}$-*martingale hazard process* of a random time $\tau$ if and only if the process $N_t - \Lambda_{t\wedge\tau}$ follows a $\mathbf{G}$-martingale, where $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$. In addition, $\Lambda_0 = 0$.

We shall first examine a special case when the $\mathbf{F}$-martingale hazard process $\Lambda$ can be expressed through a straightforward counterpart of formula (34). To this end, we introduce the following condition.

**Condition (G)** The process $F_t = P(\tau \le t \,|\, \mathcal{F}_t)$ admits a modification with increasing sample paths.

It is clear that under the hypothesis (H) the process $F$ admits a modification with increasing sample paths, so that (G) holds. Notice that $F$ is not necessarily a $\mathbf{F}$-predictable process, however. We shall study later the relation between the hypotheses (G) and (H). Also, we shall give in Section 7 an example where the hypothesis (H) (and thus also (G)) is satisfied.

**Proposition 5.3** *Assume that* (G) *holds. If the process* $\Lambda$ *given by the formula*

$$\Lambda_t = \int_{]0,t]} \frac{dF_u}{1 - F_{u-}} = \int_{]0,t]} \frac{dP(\tau \le u \,|\, \mathcal{F}_u)}{1 - P(\tau < u \,|\, \mathcal{F}_u)} \tag{54}$$

*is* $\mathbf{F}$*-predictable, then* $\Lambda$ *is the* $\mathbf{F}$*-martingale hazard process of the random time* $\tau$.

Proof: It suffices to check that $N_t - \Lambda_{t\wedge\tau}$ follows a $\mathbf{G}$-martingale, where $\mathbf{G} = \mathbf{H} \vee \mathbf{F}$. Using (43), we obtain for $t < s$

$$\begin{aligned} E(N_s - N_t \,|\, \mathcal{G}_t) &= P(t < \tau \le s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} \frac{P(t < \tau \le s \,|\, \mathcal{F}_t)}{P(\tau > t \,|\, \mathcal{F}_t)} \\ &= 1\!\!1_{\{\tau > t\}} \frac{E(F_s \,|\, \mathcal{F}_t) - F_t}{1 - F_t}. \end{aligned}$$

On the other hand, for the process $\Lambda$ given by (54) we obtain

$$E\big(\Lambda_{s\wedge\tau} - \Lambda_{t\wedge\tau}\,|\,\mathcal{G}_t\big) = E\Big(\int_{]t\wedge\tau,s\wedge\tau]} \frac{dF_u}{1-F_{u-}}\,\Big|\,\mathcal{G}_t\Big) = E(1\!\!1_{\{\tau>t\}}\,Y\,|\,\mathcal{G}_t),$$

where

$$Y \stackrel{def}{=} \int_{]t,s\wedge\tau]} \frac{dF_u}{1-F_{u-}} = 1\!\!1_{\{\tau>t\}}\,Y. \tag{55}$$

Furthermore, using (44), we get

$$E(\,1\!\!1_{\{\tau>t\}}\,Y\,|\,\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\,\frac{E(\,1\!\!1_{\{\tau>t\}}\,Y\,|\,\mathcal{F}_t)}{P(\tau>t\,|\,\mathcal{F}_t)}.$$

It is thus enough to verify that $I \stackrel{def}{=} E(\,1\!\!1_{\{\tau>t\}}\,Y\,|\,\mathcal{G}_t)$ satisfies

$$I = E(F_s\,|\,\mathcal{F}_t) - F_t. \tag{56}$$

To this end, notice that

$$\begin{aligned}
I &= E\Big(1\!\!1_{\{\tau>s\}} \int_{]t,s]} \frac{dF_u}{1-F_{u-}} + 1\!\!1_{\{t<\tau\le s\}} \int_{]t,s\wedge\tau]} \frac{dF_u}{1-F_{u-}}\,\Big|\,\mathcal{F}_t\Big)\\
&= E\Big(E\Big(1\!\!1_{\{\tau>s\}} \int_{]t,s]} \frac{dF_u}{1-F_{u-}}\Big|\mathcal{F}_s\Big) + 1\!\!1_{\{t<\tau\le s\}} \int_{]t,s\wedge\tau]} \frac{dF_u}{1-F_{u-}}\Big|\mathcal{F}_t\Big)\\
&= E\Big((1-F_s)\int_{]t,s]} \frac{dF_u}{1-F_{u-}} + \int_{]t,s]}\int_{]t,u]} \frac{dF_v}{1-F_{v-}}\,dF_u\,\Big|\,\mathcal{F}_t\Big)\\
&= E\Big((1-F_s)(\Lambda_s-\Lambda_t) + \int_{]t,s]}(\Lambda_u-\Lambda_t)dF_u\,\Big|\,\mathcal{F}_t\Big) = E(F_s-F_t\,|\,\mathcal{F}_t)
\end{aligned}$$

where the last equality can be derived along the similar lines as in the proof of part (i) in Proposition 4.4. □

The formulae established in Corollary 5.1 are still valid under the hypothesis (G). More specifically, we have the following result, which is a consequence of Corollary 5.1 combined with Proposition 5.3

**Corollary 5.3** *Suppose that* (G) *holds, and* $F_t = P(\tau \le t\,|\,\mathcal{F}_t)$ *is a continuous process. Then* $\Gamma_t = \Lambda_t = -\ln(1-F_t)$ *for every* $t \in I\!\!R_+$.
(i) *Let* $Y = h(\tau)$ *for some Borel measurable function* $h : I\!\!R_+ \to I\!\!R$. *Then*

$$E(Y\,|\,\mathcal{G}_t) = 1\!\!1_{\{\tau\le t\}}\,h(\tau) + 1\!\!1_{\{\tau>t\}}\,E\Big(\int_t^\infty h(u)e^{\Lambda_t-\Lambda_u}\,d\Lambda_u\,\Big|\,\mathcal{F}_t\Big). \tag{57}$$

(ii) *Let $Z$ be a* $\mathbf{F}$*-predictable process. Then for any* $t \le s$

$$E(Z_\tau \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau \le t\}} Z_\tau + 1\!\!1_{\{t<\tau\}} E\Big( \int_t^\infty Z_u e^{\Lambda_t - \Lambda_u}\, d\Lambda_u \,\Big|\, \mathcal{F}_t \Big),$$

$$E(Z_\tau 1\!\!1_{\{t<\tau\le s\}} \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} E\Big( \int_t^s Z_u e^{\Lambda_t - \Lambda_u}\, d\Lambda_u \,\Big|\, \mathcal{F}_t \Big).$$

### 5.3.1 Representation Theorem

We are in a position to extend Brémaud's representation theorem.

**Proposition 5.4** *Suppose that hypothesis* (G) *holds and that $F$ is continuous. Let $h$ be a* $\mathbf{F}$*-predictable process, and let* $H_t = E(h_\tau | \mathcal{G}_t)$. *Then, the process $H$ admits a decomposition in a continuous martingale and a discontinuous martingale as follows*

$$H_t = m_0 + \int_0^{t\wedge\tau} e^{\Gamma_u} dm_u + \int_{]0,t]} (h_u - H_{u-})\, dM_u, \qquad (58)$$

*where $m$ is a continuous* $\mathbf{F}$*-martingale*

$$m_t = E\Big( \int_0^\infty h_u e^{-\Gamma_u} d\Gamma_u \,\Big|\, \mathcal{F}_t \Big)$$

*and $M$ is a discontinuous* $\mathbf{G}$*-martingale, namely,* $M_t = N_t - \Gamma_{t\wedge\tau}$.

PROOF: From (57) we know that

$$\begin{aligned} H_t &= E(h_\tau \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau\le t\}} h_\tau + 1\!\!1_{\{\tau>t\}} E\Big( \int_t^\infty h_u e^{\Gamma_t - \Gamma_u} d\Gamma_u \,\Big|\, \mathcal{F}_t \Big) \\ &= 1\!\!1_{\{\tau\le t\}} h_\tau + 1\!\!1_{\{\tau>t\}} A_t. \end{aligned}$$

From the martingale representation theorem for the Brownian filtration it follows that $m$ is a continuous $\mathbf{F}$-martingale. Hence the Itô integration by parts formula yields

$$\begin{aligned} A_t &= e^{\Gamma_t} m_t - e^{\Gamma_t} \int_0^t h_u e^{-\Gamma_u} d\Gamma_u \\ &= m_0 + \int_0^t e^{\Gamma_u} dm_u + \int_0^t m_u e^{\Gamma_u} d\Gamma_u - \int_0^t h_u d\Gamma_u \end{aligned}$$

$$
\begin{aligned}
&- \int_0^t e^{\Gamma_u} \int_0^u h_v e^{-\Gamma_v} d\Gamma_v \, d\Gamma_u \\
&= m_0 + \int_0^t e^{\Gamma_u} dm_u + \int_0^t (A_u - h_u) d\Gamma_u.
\end{aligned}
$$

Furthermore (notice that the process $A$ is continuous)

$$
\mathbb{1}_{\{\tau > t\}} A_t = \int_0^{t \wedge \tau} dA_u - \mathbb{1}_{\{\tau \leq t\}} A_\tau = \int_0^{t \wedge \tau} dA_u - \mathbb{1}_{\{\tau \leq t\}} H_{\tau-}.
$$

Then, it follows that

$$
H_t = m_0 + \int_0^{t \wedge \tau} e^{\Gamma_u} dm_u + \int_0^{t \wedge \tau} (A_u - h_u) d\Gamma_u + \mathbb{1}_{\{\tau \leq t\}} (h_\tau - H_{\tau-})
$$

Since $A_u = H_u$ for $u < \tau$, we conclude that (58) is valid. □

**Remark 5.4** The process $\int_0^{t \wedge \tau} e^{\Gamma_u} dm_u$ is a **G**-martingale (see below). If **F** is a Brownian filtration, $m$ can be written as a stochastic integral with respect to the Brownian motion $W$. The process $m$ is also a **G**-martingale.

### 5.3.2 F-compensator of $F$

Assume now that either (G) is not satisfied (and thus $F$ is not necessarily an increasing process), or (G) holds (but the increasing process $F$ is not **F**-predictable). Ss the next result shows, the **F**-martingale hazard process $\Lambda$ can still be found through a suitable modification of formula (54). In fact, the foregoing results were also derived in a more general setting (see [32, 31]).

From now on, we do not need to assume that (G) holds. We write $\widetilde{F}$ to denote the **F**-compensator of the **F**-submartingale $F$. This means that $\widetilde{F}$ is the unique **F**-predictable, right-continuous, increasing process, with $\widetilde{F}_0 = 0$, and such that the process $\widetilde{M} = F - \widetilde{F}$ follows a **F**-martingale (the existence and uniqueness of $\widetilde{F}$ is a consequence of the Doob-Meyer decomposition theorem).

**Proposition 5.5** (i) *The* **F***-martingale hazard process of a random time* $\tau$ *is given by the formula*

$$\Lambda_t = \int_{]0,t]} \frac{d\widetilde{F}_u}{1 - F_{u-}}. \tag{59}$$

(ii) *If* $\widetilde{F}_t = \widetilde{F}_{t \wedge \tau}$ *for every* $t \in I\!R_+$ *(that is, the process* $\widetilde{F}$ *is stopped at* $\tau$*) then* $\Lambda = \widetilde{F}$.

Proof: It is clear that the process $\Lambda$ given by (59) is predictable. Therefore, we need only to verify that $N_t - \Lambda_{t \wedge \tau}$ follows a **G**-martingale. We proceed along the same lines as in the proof of Proposition 5.3. In the present case, it is enough to show that for any $s \geq t$

$$\widetilde{I} := E\Big( \int_{]t, s \wedge \tau]} \frac{d\widetilde{F}_u}{1 - F_{u-}} \,\Big|\, \mathcal{F}_t \Big) = E(F_s - F_t \,|\, \mathcal{F}_t) = E(\widetilde{F}_s - \widetilde{F}_t \,|\, \mathcal{F}_t),$$

where the second equality is obvious. Notice that for any **F**-predictable process $X$ we have

$$E(1\!\!1_{\{t < \tau \leq s\}} \, X_\tau \,|\, \mathcal{F}_t) = E\Big( \int_{]t,s]} X_u \, d\widetilde{F}_u \,\Big|\, \mathcal{F}_t \Big).$$

Consequently,

$$\begin{aligned}
\widetilde{I} &= E\Big( 1\!\!1_{\{\tau > s\}} \int_{]t,s]} \frac{d\widetilde{F}_u}{1 - F_{u-}} + 1\!\!1_{\{t < \tau \leq s\}} \int_{]t, s \wedge \tau]} \frac{d\widetilde{F}_u}{1 - F_{u-}} \,\Big|\, \mathcal{F}_t \Big) \\
&= E\Big( E\Big( 1\!\!1_{\{\tau > s\}} \int_{]t,s]} \frac{d\widetilde{F}_u}{1 - F_{u-}} \Big| \mathcal{F}_s \Big) + 1\!\!1_{\{t < \tau \leq s\}} \int_{]t, s \wedge \tau]} \frac{d\widetilde{F}_u}{1 - F_{u-}} \Big| \mathcal{F}_t \Big) \\
&= E\Big( (1 - F_s) \int_{]t,s]} \frac{d\widetilde{F}_u}{1 - F_{u-}} + \int_{]t,s]} \int_{]t,u]} \frac{d\widetilde{F}_v}{1 - F_{v-}} \, d\widetilde{F}_u \,\Big|\, \mathcal{F}_t \Big) \\
&= E\Big( (\Lambda_s - \Lambda_t)(1 - F_s) + \int_{]t,s]} (\Lambda_u - \Lambda_t) \, d\widetilde{F}_u \,\Big|\, \mathcal{F}_t \Big).
\end{aligned}$$

Since $\Lambda$ is **F**-predictable and $\widetilde{M}$ is a **F**-martingale, we have

$$E\Big( \int_{]t,s]} (\Lambda_u - \Lambda_t) \, d\widetilde{M}_u \,\Big|\, \mathcal{F}_t \Big) = 0.$$

Consequently, we obtain

$$\begin{aligned}\widetilde{I} &= E\Big((\Lambda_s - \Lambda_t)(1-F_s) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, d\widetilde{F}_u \,\Big|\, \mathcal{F}_t\Big) \\ &= E\Big((\Lambda_s - \Lambda_t)(1-F_s) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, d(F_u - \widetilde{M}_u) \,\Big|\, \mathcal{F}_t\Big) \\ &= E\Big((\Lambda_s - \Lambda_t)(1-F_s) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u \,\Big|\, \mathcal{F}_t\Big).\end{aligned}$$

Our goal is to show that $\widetilde{I} = E(\widetilde{F}_s - \widetilde{F}_t \,|\, \mathcal{F}_t)$. To this end, we observe that

$$\int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u = -\Lambda_t(F_s - F_t) + \int_{]t,s]} \Lambda_u\, dF_u.$$

The Itô's integration by parts formula yields

$$\int_{]t,s]} \Lambda_u\, dF_u = \Lambda_s F_s - \Lambda_t F_t - \int_{]t,s]} F_{u-}\, d\Lambda_u \tag{60}$$

since $\Lambda$ is a process of bounded variation, so that its continuous martingale part vanishes. Finally, using (59) we get

$$\int_{]t,s]} F_{u-}\, d\Lambda_u = \Lambda_s - \Lambda_t - \widetilde{F}_s + \widetilde{F}_t.$$

Combining the formulae above, we conclude that

$$(\Lambda_s - \Lambda_t)(1-F_s) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u = \widetilde{F}_s - \widetilde{F}_t. \tag{61}$$

This completes the proof of part (i) of the proposition. Assume now that $\widetilde{F}_{t\wedge\tau} = \widetilde{F}_t$ for every $t \in I\!R_+$. This means that the process $F_t - \widetilde{F}_{t\wedge\tau}$ is a $\mathbf{F}$-martingale. We wish to show that if the process $N_t - \widetilde{F}_{t\wedge\tau}$ follows a $\mathbf{G}$-martingale, that is, $E(N_s - \widetilde{F}_{s\wedge\tau} \,|\, \mathcal{G}_t) = N_t - \widetilde{F}_{t\wedge\tau}$ for $t \le s$, or equivalently, $E(N_s - N_t \,|\, \mathcal{G}_t) = E(\widetilde{F}_{s\wedge\tau} - \widetilde{F}_{t\wedge\tau} \,|\, \mathcal{G}_t)$. By virtue of (43), we have

$$E(N_s - N_t \,|\, \mathcal{G}_t) = (1 - N_t)\, \frac{E(N_s - N_t \,|\, \mathcal{F}_t)}{E(1 - N_t \,|\, \mathcal{F}_t)}. \tag{62}$$

On the other hand,

$$E(\widetilde{F}_{s\wedge\tau} - \widetilde{F}_{t\wedge\tau} \,|\, \mathcal{G}_t) \;=\; E\big(\mathbb{1}_{\{\tau > t\}}(\widetilde{F}_{s\wedge\tau} - \widetilde{F}_{t\wedge\tau}) \,|\, \mathcal{G}_t\big)$$

$$
\begin{aligned}
&= (1-N_t)\,\frac{E(\widetilde{F}_{s\wedge\tau}-\widetilde{F}_{t\wedge\tau}\,|\,\mathcal{F}_t)}{E(1-N_t\,|\,\mathcal{F}_t)}\\
&= (1-N_t)\,\frac{E(F_s-F_t\,|\,\mathcal{F}_t)}{E(1-N_t\,|\,\mathcal{F}_t)}\\
&= (1-N_t)\,\frac{E(N_s-N_t\,|\,\mathcal{F}_t)}{E(1-N_t\,|\,\mathcal{F}_t)},
\end{aligned}
$$

where the second equality follows from (42), and the third is a consequence of our assumption that the process $F_t - \widetilde{F}_{t\wedge\tau}$ is a **F**-martingale. □

**Remark 5.5** Under assumption (H), the process $\widetilde{F}$ is never stopped at $\tau$, unless $\tau$ is a **F**-stopping time. To show this assume, on the contrary, that $\widetilde{F}_t = \widetilde{F}_{t\wedge\tau}$. Under (H), the process $F_t - \widetilde{F}_{t\wedge\tau}$ is not only a **F**-martingale, but also a **G**-martingale. Since by virtue of part (ii) in Proposition 5.5 the process $N_t - \widetilde{F}_{t\wedge\tau}$ is a **G**-martingale, we conclude that $N - F$ follows a **G**-martingale. In view of the definition of $F$, the last property reads

$$E(N_s\,|\,\mathcal{G}_t) - E(E(N_s\,|\,\mathcal{F}_s)\,|\,\mathcal{G}_t) = N_t - E(N_t\,|\,\mathcal{F}_t),$$

for $t \le s$, or equivalently

$$E(N_s - N_t\,|\,\mathcal{G}_t) = E\big(E(N_s\,|\,\mathcal{F}_s)\,|\,\mathcal{G}_t\big) - E(N_t\,|\,\mathcal{F}_t) = I_1 - I_2. \tag{63}$$

Under $(H)$, we have (cf. $(H')$

$$
\begin{aligned}
I_1 &= E(P(\tau\le s\,|\,\mathcal{F}_s)\,|\,\mathcal{F}_t\vee\mathcal{H}_t)\\
&= E(P(\tau\le s\,|\,\mathcal{F}_\infty)\,|\,\mathcal{F}_t\vee\mathcal{H}_t)\\
&= E(P(\tau\le s\,|\,\mathcal{F}_\infty)\,|\,\mathcal{F}_t)
\end{aligned}
$$

since the random variable $P(\tau \le s\,|\,\mathcal{F}_\infty)$ is obviously $\mathcal{F}_\infty$-measurable, and the $\sigma$-fields $\mathcal{F}_\infty$ and $\mathcal{H}_t$ are conditionally independent given $\mathcal{F}_t$. Consequently, $I_1 = E(N_s\,|\,\mathcal{F}_t)$. We conclude that (63) can be rewritten as follows: $E(N_s - N_t\,|\,\mathcal{G}_t) = E(N_s\,|\,\mathcal{F}_t) - E(N_t\,|\,\mathcal{F}_t)$. Furthermore, applying (62) to the right-hand side of the last equality, we obtain

$$(1-N_t)\,\frac{E(N_s-N_t\,|\,\mathcal{F}_t)}{E(1-N_t\,|\,\mathcal{F}_t)} = E(N_s-N_t\,|\,\mathcal{F}_t).$$

By letting $s$ tend to $\infty$, we obtain $N_t = E(N_t\,|\,\mathcal{F}_t)$, or more explicitly, $P(\tau \le t\,|\,\mathcal{F}_t) = \mathbb{1}_{\{\tau\le t\}}$ for every $t \in I\!R_+$. This shows that $\tau$ is a **F**-stopping time.

### 5.3.3 Value of a Rebate

The theory of dual predictable projection proves that the process $\widetilde{F}$ enjoys the property that for any **F**-predictable bounded process $h$ we have

$$E(h_\tau) = E\Big(\int_0^\infty h_s\, d\widetilde{F}_s\Big).$$

Indeed, working first with elementary predictable processes, one may check that

$$E(h_\tau \,|\, \mathcal{F}_t) = E\Big(\int_0^\infty h_s d\widetilde{F}_s \,|\, \mathcal{F}_t\Big) \tag{64}$$

$$E(h_\tau \,|\, \mathcal{G}_t) = h_\tau 1\!\!1_{(\tau \le t)} + 1\!\!1_{(\tau > t)} E\Big(\int_t^\infty h_u e^{\Gamma_t} d\widetilde{F}_u \,\Big|\, \mathcal{F}_t\Big). \tag{65}$$

This property appears to be useful in the computation of the value of a rebate

$$E\big(1\!\!1_{\{\tau \le T\}}\, h_\tau\big) = E\Big(\int_0^T h_s\, d\widetilde{F}_s\Big).$$

## 5.4 Brownian Motion Case

In all results of this section we assume that the filtration **F** is the natural filtration of some Brownian motion. In this case, the decomposition of any **F**-martingale in the filtration **G** is known up to time $\tau$ (see Jeulin's and Yor's papers [32, 31, 59]). For example, if $W$ is a Brownian motion with respect to **F**, its decomposition in the filtration **G**, before the default time $\tau$, reads

$$W_{t\wedge\tau} = \widetilde{W}_{t\wedge\tau} - \int_0^{t\wedge\tau} \frac{d\langle W, F\rangle_s}{1 - F_{s-}},$$

where $(\widetilde{W}_{t\wedge\tau}\,; t \ge 0)$ is a continuous **G**-martingale with the increasing process $t \wedge \tau$. We may thus interpret this martingale as a **G**-Brownian motion stopped at time $t$. If the dynamics of a value of the firm $V$ are given by

$$dV_t = V_t(r_t\, dt + \sigma_t\, dW_t)$$

in a default-free framework, where $W$ is a Brownian motion under the e.m.m., the decomposition of $V$ with respect to the enlarged filtration

$\mathbf{G}$ is

$$dV_t = V_t\big(r_t\,dt - \sigma_t\,\frac{d\langle W,F\rangle_t}{1-F_{t-}} + \sigma_t\,d\widetilde{W}_t\big)$$

provided that we restrict our attention to times before default. Therefore, the default acts as a change of drift term in the dynamics of the price process. Some examples will be given in Section 6.

In some examples, $F$ is a continuous increasing process. In this case, the bracket $\langle W,F\rangle$ is equal to zero, and the $\mathbf{F}$-Brownian motion $W$ remains a Brownian motion with respect to the filtration $\mathbf{G}$ up to time $\tau$. Therefore, any $\mathbf{F}$-martingale is equal to a $\mathbf{G}$-martingale up to time $\tau$. Moreover, the hazard process and the intensity process are equal.

### 5.4.1 Representation Theorem

The filtration $\mathbf{F}$ is assumed here to be the natural filtration of some Brownian motion. On the other hand, in contrast to Proposition 5.4, we no longer postulate that the hypothesis (G) is satisfied. On the other hand, we still make an assumption that the process $F$ is continuous. Under this set of assumptions, we get the following representation theorem.

**Proposition 5.6** *Let $h$ be a $\mathbf{F}$-predictable process, and $H_t = E(h_\tau \,|\, \mathcal{G}_t)$. Then, the process $H$ admits a decomposition in a continuous martingale and a discontinuous martingale as follows*

$$H_t = H_0 + \int_0^{t\wedge\tau} \Phi_u d\widehat{B}_u + \int_{]0,t]} (h_u - H_{u-})\,dM_u, \tag{66}$$

*where $\widehat{B}$ is a $\mathbf{G}$-Brownian motion and $M$ is the discontinuous $\mathbf{G}$-martingale, $M_t = N_t - \Gamma_{t\wedge\tau}$.*

Proof: From (65) we know that

$$\begin{aligned} H_t &= E(h_\tau \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau\le t\}} h_\tau + \mathbb{1}_{\{\tau>t\}} E\Big(\int_t^\infty h_u e^{\Gamma_t} d\widetilde{F}_u \,\Big|\, \mathcal{F}_t\Big) \\ &= \mathbb{1}_{\{\tau\le t\}} h_\tau + \mathbb{1}_{\{\tau>t\}} A_t. \end{aligned}$$

The process $\nu \stackrel{def}{=} F - \widetilde{F}$ is a $\mathbf{F}$ martingale. Therefore, the process $A$ can be written as

$$A_t = e^{\Gamma_t} E\Big(\int_t^\infty h_u d\widetilde{F}_u \,\Big|\, \mathcal{F}_t\Big) = e^{\Gamma_t} E\Big(\int_t^\infty h_u dF_u \,\Big|\, \mathcal{F}_t\Big)$$

$$= e^{\Gamma_t} m_t - e^{\Gamma_t} \int_0^t h_u dF_u$$

where $m_t = E\Big( \int_0^\infty h_u dF_u \,\Big|\, \mathcal{F}_t \Big)$. Hence the Itô integration by parts formula yields

$$\begin{aligned} A_t &= m_0 + \int_0^t e^{\Gamma_u} (dm_u - h_u dF_u) \\ &\quad + \int_0^t A_u e^{\Gamma_u} (dF_u + e^{\Gamma_u} d\langle F \rangle_u) - \int_0^t d\langle m - \int_0^\cdot h_u dF_u, e^{\Gamma} \rangle_s . \end{aligned}$$

Let us compute the bracket

$$\begin{aligned} d\langle m - \int_0^\cdot h_u dF_u, e^{\Gamma} \rangle_s &= e^{\Gamma_s} d\langle m - \int_0^\cdot h_u dF_u, \Gamma \rangle_s \\ &= e^{2\Gamma_s} \big( d\langle m, F \rangle_s - h_s d\langle F \rangle_s \big). \end{aligned}$$

It follows that

$$\begin{aligned} A_t &= m_0 + \int_0^t e^{\Gamma_u} \big( dm_u + e^{\Gamma_u} d\langle m, F \rangle_u \big) \\ &\quad + \int_0^t e^{\Gamma_u} (A_u - h_u) \big( dF_u + e^{\Gamma_u} d\langle F \rangle_u \big). \end{aligned}$$

The processes

$$\widetilde{m}_t = m_t + \int_0^t e^{\Gamma_u} d\langle m, F \rangle_u, \quad \widetilde{\nu}_t = \nu_t + \int_0^t e^{\Gamma_u} d\langle \nu \rangle_u$$

are $\mathbf{G}$-martingales, and they can be written in terms of $\widetilde{B}_{t\wedge\tau} = B_{t\wedge\tau} + \int_0^{t\wedge\tau} \dfrac{d\langle B, F \rangle_s}{1 - F_s}$. Finally, we have

$$\begin{aligned} H_t &= (h_\tau - H_{\tau-}) \mathbb{1}_{\{\tau \le t\}} + m_0 \\ &+ \int_0^{t\wedge\tau} e^{\Gamma_u} \left( d\widetilde{m}_u + (H_u - h_u) d\widetilde{\nu}_u \right) + \int_0^{t\wedge\tau} (H_u - h_u) e^{\Gamma_u} dA_u . \end{aligned}$$

This completes the proof. □

### 5.4.2 Relationship Between Hypotheses (G) and (H)

It appears that models in which hypothesis (G) holds are close to that in which (H) is satisfied. More precisely, we have the following result.

**Proposition 5.7** *Suppose that the process $F$ is continuous. The two following conditions are equivalent:*
(a) *the process $F_t = P(\tau \le t \,|\, \mathcal{F}_t)$ is increasing,*
(b) *if $(Y_t, t \ge 0)$ is a* $\mathbf{F}$*-martingale, then $(Y_{t\wedge\tau}, t \ge 0)$ is a* $\mathbf{G}$*-martingale.*

Proof: If (G) holds, then the process $K_t = E(\mathbb{1}_{\{\tau > t\}} \,|\, \mathcal{F}_t)$ is decreasing. Therefore, the $\mathbf{F}$-Brownian motion $W$ remains a $\mathbf{G}$-Brownian motion up to time $\tau$, and thus (b) holds.

If (b) holds, the bracket $\langle W, F\rangle$ is equal to zero. This implies that the martingale part of $F$ is equal to zero, therefore the supermartingale $F$ is an increasing process. □

## 5.5 Uniqueness of a Martingale Hazard Process $\Lambda$

We shall first examine the relationship between the concept of a $\mathbf{F}$-martingale hazard process $\Lambda$ of $\tau$ and the classic notion of $\mathbf{G}$-*compensator* of $\tau$ (or a dual predictable projection of the associated first jump process $N$).

**Definition 5.3** A process $A$ is a $\mathbf{G}$-*compensator* of $\tau$ if and only if the following conditions are satisfied: (i) $A$ is a $\mathbf{G}$-predictable right-continuous increasing process, with $A_0 = 0$, (ii) the process $N - A$ is a $\mathbf{G}$-martingale.

It is well known that for any random time $\tau$ and any filtration $\mathbf{G}$ such that $\tau$ is a $\mathbf{G}$-stopping time there exists a unique $\mathbf{G}$-compensator $A$ of $\tau$. Moreover, $A_t = A_{t\wedge\tau}$, that is, $A$ is stopped at $\tau$. In the next auxiliary result, we deal with an arbitrary filtration $\mathbf{F}$ which, when combined with the natural filtration $\mathbf{H}$ of a $\mathbf{G}$-stopping time $\tau$, generates the enlarged filtration $\mathbf{G}$.

**Lemma 5.5** *Let* $\mathbf{F}$ *be an arbitrary filtration*[7] *such that* $\mathbf{G} = \mathbf{H} \vee \mathbf{F}$.
(i) *Let $A$ be a* $\mathbf{G}$*-predictable right-continuous increasing process satisfy-*

[7] As usual, we assume that $\mathbf{F}$ satisfies the 'usual conditions.'

*ing* $A_t = A_{t\wedge\tau}$. *Then there exists a* **F***-predictable right-continuous increasing process* $\Lambda$ *such that* $A_t = \Lambda_{t\wedge\tau}$. (ii) *Let* $\Lambda$ *be a* **F***-predictable right-continuous increasing process. Then* $A_t = \Lambda_{t\wedge\tau}$ *is a* **G***-predictable right-continuous increasing process.*

The next proposition summarizes the relationships between the **G**-compensator of $\tau$ and the **F**-martingale hazard process $\Lambda$ of $\tau$. Once again, **F** is an arbitrary filtration such that $\mathbf{G} = \mathbf{H} \vee \mathbf{F}$.

**Proposition 5.8** (i) *Let* $\Lambda$ *be a* **F***-martingale hazard process of* $\tau$. *Then the process* $A_t = \Lambda_{t\wedge\tau}$ *is the* **G***-compensator of* $\tau$. (ii) *Let* $A$ *be the* **G***-compensator of* $\tau$. *Then there exists a* **F***-martingale hazard process* $\Lambda$ *such that* $A_t = \Lambda_{t\wedge\tau}$.

Let us recapitulate the results above. First, for any random time $\tau$ on some probability space $(\Omega, \mathcal{G}, P)$, and an arbitrary filtration **F** there exists a **F**-martingale hazard process $\Lambda$ of $\tau$. Furtheremore, it is unique up to time $\tau$, in the following sense: if $\Lambda^1$ and $\Lambda^2$ are two **F**-martingale hazard processes of $\tau$, then $\Lambda^1_{t\wedge\tau} = \Lambda^2_{t\wedge\tau}$. To ensure the uniqueness after $\tau$ of a **F**-martingale hazard processes we need to impose additional restrictions on $\Lambda$.

Assume now that we are given a **G**-stopping time $\tau$ for some filtration **G**. Then there exist several distinct filtrations **F** such that $\mathbf{G} = \mathbf{H} \vee \mathbf{F}$. Assume that $\mathbf{G} = \mathbf{H} \vee \mathbf{F}^1 = \mathbf{H} \vee \mathbf{F}^2$, and denote by $\Lambda^i$ a $\mathbf{F}^i$-martingale hazard process of $\tau$. Then $\Lambda^1_{t\wedge\tau} = A_{t\wedge\tau} = \Lambda^2_{t\wedge\tau}$. It seems reasonable to search for the $\hat{\mathbf{F}}$-martingale hazard process where $\hat{\mathbf{F}}$ is a 'minimal' filtration such that $\mathbf{G} = \mathbf{H} \vee \hat{\mathbf{F}}$.

## 5.6 Relationship Between Hazard Processes $\Gamma$ and $\Lambda$

Let us assume that $\Gamma$ is well defined (in particular, $\tau$ is not a **F**-stopping time). Under assumption (G), if the **F**-hazard process $\Gamma$ and the **F**-martingale hazard process $\Lambda$ are continuous processes then obviously $\Gamma_t = \Lambda_t = -\ln(1 - F_t)$. More precisely, under (G), the continuity of $\Gamma$ implies the continuity of $\Lambda$ and the validity of (54). On the other hand, if the **F**-martingale hazard process $\Lambda$ is given by (54), and $\Lambda$ is a discontinuous process, then necessarily $\Lambda \neq \Gamma$.

Recall that, if the **F**-hazard process $\Gamma$ is well defined, then for any $\mathcal{F}_s$-measurable random variable $Y$ we have (cf. (45))

$$E(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} E\big(Y e^{\Gamma_t - \Gamma_s} \,|\, \mathcal{F}_t\big). \tag{67}$$

The natural question which arises is this context is whether we may substitute $\Gamma$ with the **F**-martingale hazard function $\Lambda$ in the formula above. Of course, the answer is trivial (and positive) when it is known that equality $\Lambda = \Gamma$ is valid. For instance, this holds if condition (G) is satisfied and $\Gamma$ is a continuous process (recall that in this case $\Lambda_t = \Gamma_t = -\ln(1-F_t)$). If, in addition, the process $\Lambda = \Gamma$ is absolutely continuous then for any $\mathcal{F}_s$-measurable random variable $Y$

$$E(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} E\big(Y e^{-\int_t^s \lambda_u \, du} \,|\, \mathcal{F}_t\big). \tag{68}$$

If the **F**-hazard process $\Gamma$ exists, but follows a discontinuous process then the equality $\Lambda = \Gamma$ is never satisfied. More precisely, under (G), we have

$$e^{-\Gamma_t} = e^{-\Lambda_t^c} \prod_{0\le u\le t} (1 - \Delta\Lambda_u), \tag{69}$$

where $\Lambda^c$ is the continuous component of $\Lambda$, that is,

$$\Lambda_t^c = \Lambda_t - \sum_{0\le u\le t} \Delta\Lambda_u.$$

The next result covers also the case when the **F**-hazard process $\Gamma$ does not exist (for instance, when $\tau$ is a **F**-stopping time). We shall work directly with the **F**-martingale hazard process $\Lambda$. It appears that equality (67) remains valid with $\Gamma$ replaced by $\Lambda$, provided that an additional continuity condition is satisfied. Notice that for the process $V$ in formula (70) below to be well defined, we need to specify the **F**-martingale hazard process $\Lambda$ not only up to $\tau$, but also after $\tau$. The proof of the next result is almost identical to that of Proposition 3.2.

**Proposition 5.9** *Assume that the hypothesis* (H) *is valid and the* **F**-*martingale hazard process* $\Lambda$ *of* $\tau$ *is a continuous process. For a fixed* $s > 0$, *let* $Y$ *be a* $\mathcal{G}_s$-*measurable random variable. If the* (*right-continuous*) *process* $V$, *given by the formula*

$$V_t \stackrel{def}{=} E\big(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{F}_t\big), \quad \forall\, t \in [0, s], \tag{70}$$

*is continuous at* $\tau$, *that is,* $\Delta V_{s\wedge\tau} = V_{s\wedge\tau} - V_{(s\wedge\tau)-} = 0$, *then for any* $t < s$ *we have*

$$E(\mathbb{1}_{\{\tau>s\}} Y \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} E\big(Y e^{\Lambda_t - \Lambda_s} \mid \mathcal{G}_t\big). \tag{71}$$

**Remark 5.6** Let us restrict our attention to the case where $\mathbf{F}$ is a Brownian filtration, as it is the case in these notes. Then

$$E\big(Y e^{\Lambda_t - \Lambda_s} \mid \mathcal{F}_t\big) = e^{\Lambda_t} E\big(Y e^{-\Lambda_s} \mid \mathcal{F}_t\big).$$

The martingale $E\big(Y e^{-\Lambda_s} \mid \mathcal{F}_t\big)$ is continuous as any Brownian martingale and $e^{\Lambda_t}$ is continuous by hypothesis. Therefore $V$ is a continuous process.

The following corollary provides a sufficient condition for the martingale hazard process $\Lambda$ to determine the conditional survival probability of $\tau$ given the $\sigma$-field $\mathcal{G}_t$.

**Corollary 5.4** *Assume that the hypothesis* (H) *is valid. Let the* $\mathbf{F}$-*martingale hazard process* $\Lambda$ *of* $\tau$ *be a continuous process. For a fixed* $s > 0$, *if the process* $V$ *given by the formula*

$$V_t = E\big(e^{\Lambda_t - \Lambda_s} \mid \mathcal{F}_t\big), \quad \forall\, t \in [0, s], \tag{72}$$

*is continuous at* $\tau$, *that is,* $\Delta V_{s\wedge\tau} = V_{s\wedge\tau} - V_{(s\wedge\tau)-} = 0$, *then for any* $t \le s$ *we have*

$$P(\tau > s \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} E\big(e^{\Lambda_t - \Lambda_s} \mid \mathcal{F}_t\big). \tag{73}$$

# 6 Case where $\tau$ is a $\mathcal{F}_\infty$-measurable variable

Our aim is to study some particular examples, in which the hypothesis (G) is not satisfied. Another example can be found in Kusuoka [33]. As in Section 5.4, we assume throughout that $\mathbf{F}$ is the Brownian filtration. We shall mainly study the last passage times.

## 6.1 Notation and Basic Results

Suppose that the dynamics of the value of a firm are

$$dV_t = \mu(V_t)\, dt + \sigma(V_t)\, dW_t, \quad V_0 = v > 0 \tag{74}$$

where $\sigma > 0$ and that the interest rate $r$ is equal to 0. In what follows, a level $a$ is considered. We denote by $\tau_a(V)$ the first time when $V$ is equal to $a$:

$$\tau_a(V) = \inf\{\, t \geq 0 \,:\, V_t = a\}$$

with $\inf\{\emptyset\} = +\infty$. When $V$ is a transient diffusion, we introduce $\gamma_a(V)$ the last time where $V$ is equal to $a$, that is,

$$\gamma_a(V) = \sup\{t \geq 0 \,:\, V_t = a\}.$$

If there is no ambiguity, we shall write briefly $\tau_a = \tau_a(V)$, $\gamma_a = \gamma_a(V)$. For a fixed $t$, we shall also use the following random times

$$\begin{aligned} g_t^a(V) &= \sup\{\, s \leq t \,:\, V_s = a\}, \\ d_t^a(V) &= \inf\{\, u \geq t \,:\, V_u = a\}. \end{aligned}$$

The random time $g_t^a$ is the left extremity of the excursion which straddles over $t$, or the last passage at the level $a$ before time $t$. The random time $d_t^a$ is the first passage at the level $a$ after time $t$. The random times $g_t^a$ and $\gamma_a$ are obviously $\mathcal{F}_\infty$-measurable; they are not stopping times, however. On the other hand, the random times $\tau_a$ and $d_t^a$ are $\mathbf{F}$-stopping times. Let $u < t$. The equality of events

$$\{g_t^a(V) \leq u\} = \{\tau_a(V) \leq u\} \cap \{d_u^a > t\}$$

will be of constant use.

In the case $\mu = 0$ and $\sigma = 1$, the process $V$ is equal to $v + W$, where $W$ is a Brownian motion and using results of Section 2.1.1 we get

$$\begin{aligned} d_t^a(V) &= t + \inf\{u \geq 0 \,:\, W_{u+t} - W_t = \alpha - W_t\} \\ &= t + \widehat{\tau}_{\alpha - W_t} \overset{law}{=} t + \frac{(\alpha - W_t)^2}{G^2}. \end{aligned} \tag{75}$$

Here $\alpha = a - v$ and $\widehat{\tau}_b = \inf\{u \geq 0 \,:\, \widehat{W}_u = b\}$, where $\widehat{W}$ is a Brownian motion independent of $\mathcal{F}_t$, and $G$ is a Gaussian variable, with mean 0 and variance 1, independent of $W$.

Similar studies can be done for the process

$$d\widetilde{V}_t = \widetilde{V}_t(\mu\, dt + \sigma\, dW_t), \quad \widetilde{V}_0 = v,$$

thanks to the following relationships

$$\begin{aligned}
g^a(\widetilde{V}) &= \sup\{t \le 1 \,:\, (\mu - \frac{\sigma^2}{2})t + \sigma W_t = \ln\frac{a}{v}\,\} \\
&= \sup\{t \le 1 \,:\, \nu t + W_t = \alpha\,\} \\
&= \sup\{t \le 1 \,:\, \widetilde{W}_t = \alpha\,\} = g^{\alpha}(\widetilde{W}), \\
\gamma_a(\widetilde{V}) &= \sup\{t \,:\, \widetilde{W}_t = \alpha\,\},
\end{aligned}$$

where

$$\nu = \frac{1}{\sigma}(\mu - \frac{\sigma^2}{2}), \quad \alpha = \frac{1}{\sigma}\ln\frac{a}{v} < 0, \quad \widetilde{W}_t = \nu t + W_t\,.$$

### 6.1.1 Valuation of Defaultable Claims

For simplicity, we assume that $r = 0$. Our aim is to compute the value of a contingent claim with payoff $G(V_T)$, where $T$ is a fixed time, if the default has not appeared before time $T$. This payoff is made either at time $T$ or later (see below for details). The case where a payment of $h_\tau$ is made at time $\tau$ if the default has appeared before $T$, and where $h$ is some given **F**-predictable process is also taken into account. In this setting, the value of the defaultable claim is the expectation of

$$G(V_T)1\!\!1_{\{T<\tau\}} + h_\tau 1\!\!1_{\{\tau\le T\}}\,.$$

The value of the defaultable claim for a **G**-informed agent consists of two components:

• **Terminal payoff**

The value of the terminal payoff is

$$\begin{aligned}
E(G(V_T)1\!\!1_{\{T<\tau\}}\,|\,\mathcal{G}_t) &= 1\!\!1_{\{\tau>t\}}\frac{E(G(V_T)1\!\!1_{\{T<\tau\}}\,|\,\mathcal{F}_t)}{E(1\!\!1_{\{\tau>t\}}\,|\,\mathcal{F}_t)} \\
&= 1\!\!1_{\{\tau>t\}}E(G(V_T)e^{\Gamma_t-\Gamma_T}\,|\,\mathcal{F}_t)
\end{aligned}$$

where $e^{-\Gamma_t} = 1 - F_t = P(\tau > t\,|\,\mathcal{F}_t)$. In the special case when $e^{-\Gamma_t} = \psi(V_t)$, we obtain, due to Markov property of $V$

$$E(G(V_T)e^{-\Gamma_T}\,|\,\mathcal{F}_t) = \Psi(V_t, T-t)$$

with $\Psi(x,u) = E\big(G(V_u^x)\psi(V_u^x)\big)$, where $V^x$ is the solution of (74) with the initial value $x$. In order to evaluate this part, the computation of the intensity is not sufficient.

• **Rebate**

As we mentioned earlier, if $h$ is a (bounded) **F**-predictable process then

$$E\big(h_\tau 1\!\!1_{\{t<\tau\le T\}} | \mathcal{G}_t\big) = 1\!\!1_{\{t<\tau\}} e^{\Gamma_t} E\Big(\int_t^T h_s \, d\widetilde{F}_s | \mathcal{F}_t\Big).$$

As we shall see in what follows, in many examples the process $\widetilde{F}$ evolves as a local time $\ell^a(V)$. In the particular case when $h_s = h(V_s)$, the computation of the rebate part can be done using the well known property of the local time: $\int_0^T h(V_s)\, d\ell_s^a(V) = h(a)\ell_T^a(V)$.

### 6.1.2 Scale Function

Recall that a *scale function* for a diffusion $V$ is a function $s : I\!\!R \to I\!\!R$ such that the process $s(V)$ follows a local martingale. In our setting, the scale function for $V$ is known to satisfy $s'(x) = c \exp\big(-2\int_0^x \mu(y)\sigma^{-2}(y)\,dy\big)$, where $c$ is a constant. In the case of constant coefficients $\mu$ and $\sigma$, with $\mu < 0$ and $\sigma > 0$, we may choose $s(x) = \exp\big(-2\mu\sigma^{-2}x\big)$, so that $s$ is a strictly increasing function and $s(-\infty) = 0$.

Let $\ell^a(V)$ stand for the local time of $V$ at the level $a$ and $L^y(Y)$ the local time of $Y = s(V)$ at the level $y$ (of course, both $V$ and $s(V)$ are continuous semimartingales). The density of occupation time formula for $Y$ leads to

$$\begin{aligned}
\int_{I\!\!R} f(y) L_t^y(Y)\, dy &= \int_0^t f(Y_s)\, d\langle Y\rangle_s = \int_0^t f(s(V_s))\, [s'(X_s)]^2 \, d\langle X\rangle_s \\
&= \int_{I\!\!R} f(s(a))\, [s'(a)]^2 \ell_t^a(V)\, da \\
&= \int_{I\!\!R} f(y) s'(s^{-1}(y))\, \ell_t^{s^{-1}(y)}(V)\, dy.
\end{aligned}$$

This shows that $L_t^{s(a)}(Y) = s'(s(a))\, \ell_t^a(V)$.

## 6.2 Last Passage Time of a Transient Diffusion

Let $V$ be a transient diffusion and $s$ a scale function such that $s(-\infty) = 0$ and $s(x) > 0$. Let $\gamma_a \stackrel{def}{=} \sup\{t \geq 0 : V_t = a\}$ be the last passage time of $V$ at the level $a$. The value of the firm will never be at $a$ after time $\gamma_a$.

### 6.2.1 F-compensator of $N$

We shall now focus on the explicit evaluation of the **F**-compensator of the first jump process associated with the last passage time of a transient diffusion.

**Lemma 6.1** *We have*

$$e^{-\Gamma_t} = P(\gamma_a > t \,|\, \mathcal{F}_t) = \frac{s(V_t)}{s(a)} \wedge 1.$$

*The* **F**-*compensator of the process* $N_t = \mathbb{1}_{\{\gamma_a \leq t\}}$ *is* $\widetilde{F}_t = \dfrac{1}{2s(a)} L_t^{s(a)}$, *where* $L$ *is the local time of the continuous semimartingale* $s(V)$.

PROOF: These results are well known (see, for example, Yor [59], p.48). We reproduce here the proof. Observe that

$$\begin{aligned} P_v\big(\gamma_a > t \,|\, \mathcal{F}_t\big) &= P_v\Big(\inf_{u \geq t} V_u < a \,\Big|\, \mathcal{F}_t\Big) \\ &= P_v\Big(\sup_{u \geq t}(-s(V_u)) > -s(a) \,\Big|\, \mathcal{F}_t\Big) \\ &= P_{V_t}\Big(\sup_{u \geq 0}(-s(V_u)) > -s(a)\Big) = \frac{s(V_t)}{s(a)} \wedge 1, \end{aligned}$$

where the two equalities follow from the Markov property of $V$, and the fact that if $M$ is a continuous local martingale with $M_0 = 1$, $M_t \geq 0$, and $\lim_{t \to \infty} M_t = 0$, then

$$\sup_{t \geq 0} M_t \stackrel{law}{=} \frac{1}{U},$$

where $U$ has a uniform law on $[0,1]$ (see Revuz and Yor [47], Chap.2, Ex.3.12). Tanaka's formula implies that

$$\frac{s(V_t)}{s(a)} \wedge 1 = M_t - \frac{1}{2s(a)} L_t^{s(a)}$$

where $M$ is a martingale and the needed result is then easily obtained. $\square$

### 6.2.2 Application to the Valuation of Defaultable Claims

In this section, we shall explicitly compute the value of a defaultable claim in the particular case where $V_t = v + \mu t + \sigma W_t$, with $\mu < 0$. In this case, the scale function is $s(x) = \exp(-2\mu\sigma^{-2}x)$. Recall that we consider here the default time $\tau = \gamma_a \stackrel{def}{=} \sup\{t \geq 0 \,:\, V_t = a\}$. Note that $z \wedge 1 = 1 - (1-z)^+$. Therefore, $E(h(V_T)Z_T \,|\, \mathcal{F}_t) = \Psi(V_t, T-t)$, where

$$\Psi(x,u) = E\left[h(x+\mu u + \sigma W_u)\left(1 - \left(1 - \frac{s(x+\mu u+\sigma W_u)}{s(a)}\right)^+\right)\right],$$

or equivalently,

$$\Psi(x,u) = E[h(V_u^x)] - E\left[h(V_u^x)\left(1 - \frac{1}{s(a)} \exp -\frac{2\mu}{\sigma}(x+\mu u + \sigma W_u)\right)^+\right],$$

where $V_u^x = x + \mu u + \sigma W_u$. In the case of a defaultable zero-coupon bond with zero recovery, the computation reduces to the classic case of a European option. Indeed, from the above computations

$$E(\mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{s(a) - E((s(a) - s(V_T))^+ \,|\, \mathcal{F}_t)}{s(V_t) \wedge s(a)}$$

and the calculation of

$$E\big((s(a) - s(V_T))^+ \,|\, \mathcal{F}_t\big) = E\Big(\big(s(a) - \exp\big[-\frac{2\mu}{\sigma^2}(v + \mu T + \sigma W_T)\big]\big)^+ \,\Big|\, \mathcal{F}_t\Big)$$

is the same as in the case of a put option in the Black-Scholes model. In the other cases, even though the computations are sometimes heavy, they only involve the Gaussian law.

## 6.3 Last Passage Time Before Bankruptcy

We now assume that $\tau = g^a_{\tau_0}(V) = \sup\{t \le \tau_0 \,:\, V_t = a\}$, where

$$\tau_0 = \tau_0(V) = \inf\{t \ge 0 \,:\, V_t = 0\}$$

is the bankruptcy date, and $v > a$. Let $\mu = 0$ and let $\sigma$ be a nonnegative constant. Then

$$P\big(g^a_{\tau_0}(V) \le t \,|\, \mathcal{F}_t\big) = P\big(d^\alpha_t(W) > \tau_0(W) \,|\, \mathcal{F}_t\big)$$

on the set $\{t < \tau_0(W)\}$, where we set $\alpha = \dfrac{v-a}{\sigma}$. It is easy to prove that

$$P\big(d^\alpha_t(W) < \tau_0(W) \,|\, \mathcal{F}_t\big) = \Phi(\sigma W_{t\wedge\tau_0(W)}),$$

where the function $\Phi : I\!R \to I\!R$ equals

$$\begin{array}{rcll} \Phi(x) = P_x(\tau_\alpha(W) < \tau_0(W)) & = & x/\alpha & \text{for} \quad 0 < x < \alpha, \\ & = & 1 & \text{for} \quad \alpha < x, \\ & = & 0 & \text{for} \quad x < 0. \end{array}$$

Consequently, on the set $\{\tau_0(V) > t\}$ we have

$$E\big(g^a_{\tau_0} \le t \,|\, \mathcal{F}_t\big) = \frac{(\alpha - W_{t\wedge\tau_0})^+}{\alpha} = \frac{(\alpha - W_t)^+}{\alpha}.$$

We deduce that the $\mathbf{F}$-compensator of the process $N_t = 1\!\!1_{\{g^a_{\tau_0} \le t\}}$ is $A_t = \dfrac{1}{2\alpha} L^\alpha_t(W)$.

## 6.4 Last Passage Time Before Maturity

In this section, we assume that a firm operates until time $T + \varepsilon = \theta$, where $T$ is a fixed time, and it promises to pay to the investors at date $\theta$ the amount $H(V_T)$. If the value of the firm remains between $T$ and $\theta$ above a level $a$, for some $a < v$, then the firm defaults and the payment of $H(V_T)$ at time $\theta$ is not made. In this setting, the value of the defaultable claim with a rebate given by the process $h_t$ is the expectation of

$$H(V_T) 1\!\!1_{\{T<\tau\}} + H(V_T) 1\!\!1_{\{V_T \ge a\}} 1\!\!1_{\{\tau \le T\}} + h_\tau 1\!\!1_{\{V_T \le a\}} 1\!\!1_{\{\tau \le T\}},$$

where

$$\tau = \sup\{\, t \le \theta \,:\, V_t = a\} = g^a_\theta(V).$$

If $V_t > a$ for every $t \le \theta$, we set $\tau = \theta$. If the time $\varepsilon$, which acts as a delay, is equal to 0, the problem is easier, at least at time 0, since in this case the payoff is a $\mathcal{F}_1$-measurable claim $H(V_1)\mathbb{1}_{\{V_1 > a\}} + h_g \mathbb{1}_{\{V_1 < a\}}$, and we are reduced to a "digital" computation for the first part (we shall explain later how to evaluate the rebate part). For simplicity, we set $\theta = 1$ in what follows.

### 6.4.1 Brownian Motion Case

We present here the case of a Brownian motion. If $V$ is a Brownian motion with constant drift, the calculations can also be done, but they are more involved.

**• Computation of the intensity function**

Working in the **H**-filtration is easy, since we need only to know the probability law of the random variable $g = \sup\{t \le 1 \,:\, W_t = 0\}$, where the Brownian motion $W$ starts from 0. It is well known that $g$ follows a arcsine law, i.e.,

$$F(t) = P(g \le t) = \frac{2}{\pi}\mathrm{Arcsin}\sqrt{t}.$$

Therefore, the intensity function of the random time $g$ is

$$\lambda(t) = \frac{f(t)}{1 - F(t)} = \frac{1}{(1 - \mathrm{Arcsin}(\sqrt{t}))\sqrt{t(1-t)}}.$$

**• Computation of the F-compensator**

**Lemma 6.2** *The* **F***-hazard process of* $N$ *is*

$$F_t = P(g \le t \,|\, \mathcal{F}_t) = \Phi\left(\frac{|\,W_t\,|}{\sqrt{1-t}}\right) \qquad (76)$$

*where*

$$\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x \exp(-\frac{u^2}{2})\, du.$$

*The* $\mathbf{F}$*-compensator of* $N_t = 1\!\!1_{\{g \le t\}}$ *equals*

$$\widetilde{F}_t = \int_0^t \sqrt{\frac{2}{\pi}}\, \frac{dL_s}{\sqrt{1-s}},$$

*where* $L$ *is the local time at level 0 of the Brownian motion* $W$*.*

PROOF: This result is well known and can be found in Yor [59]. We reproduce the proof. For $t < 1$, the set $\{g \le t\}$ is equal to $\{d_t > 1\}$. The result follows from (75) and the following equality[8]

$$P\Big(\frac{a^2}{G^2} > 1 - t\Big) = \Phi\Big(\frac{|\,a\,|}{\sqrt{1-t}}\Big).$$

Then, the Itô-Tanaka formula combined with the identity $x\Phi'(x) + \Phi''(x) = 0$ lead to

$$\begin{aligned}
P(g \le t \,|\, \mathcal{F}_t) &= \Phi\left(\frac{|\,W_t\,|}{\sqrt{1-t}}\right) \\
&= \int_0^t \Phi'\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right) d\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right) + \frac{1}{2}\int_0^t \frac{ds}{1-s}\Phi''\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right) \\
&= \int_0^t \Phi'\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right) \frac{\mathrm{sgn}(W_s)}{\sqrt{1-s}}\, dW_s + \int_0^t \frac{dL_s}{\sqrt{1-s}}\Phi'\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right) \\
&= \int_0^t \Phi'\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right) \frac{\mathrm{sgn}(W_s)}{\sqrt{1-s}}\, dW_s + \sqrt{\frac{2}{\pi}}\int_0^t \frac{dL_s}{\sqrt{1-s}}\,.
\end{aligned}$$

It follows that the $\mathbf{F}$-compensator of $N$ is

$$\widetilde{F}_t = \sqrt{\frac{2}{\pi}} \int_0^{t\wedge g} \frac{dL_s}{\sqrt{1-s}}.$$

Therefore, since $\widetilde{F}_t = \widetilde{F}_{t\wedge g}$ the $\mathbf{F}$-martingale hazard process of $g$ is $d\Lambda_t = \sqrt{\frac{2}{\pi}}\, \frac{dL_t}{\sqrt{1-t}}$. □

**Remark 6.1** (i) In this particular case, it is possible to give the decomposition of the Brownian motion in the enlarged filtration, namely,

$$W_t = \widetilde{\widetilde{W}}_t \quad - \quad \int_0^t 1\!\!1_{[0,g]}(s)\, \frac{\Phi'}{1-\Phi}\left(\frac{|\,W_s\,|}{\sqrt{1-s}}\right) \frac{\mathrm{sgn}(W_s)}{\sqrt{1-s}}\, ds$$

[8] As usual, $G$ stands for a standard Gaussian r.v.

$$+ \quad \mathbb{1}_{\{g \le t\}} \operatorname{sgn}(W_1) \int_g^t \frac{\Phi'}{\Phi} \left( \frac{|W_s|}{\sqrt{1-s}} \right) ds.$$

(ii) It is interesting to notice that the continuous **F**-submartingale $F_t = P(g \le t \,|\, \mathcal{F}_t)$ does not follow an increasing process, so that $F_t \ne 1 - \exp(-\Lambda_t)$, and we have an example when the equality $\Gamma = \Lambda$ does not hold. In order to ensure that the conditional expectation follows an increasing process, it is necessary to work with the filtration $\widetilde{\mathbf{F}}$ generated by the **G**-Brownian motion $\widetilde{B}$ (See Azéma et al. [2] or Yor [59], Prop. 14.5). No financial interpretation of this new filtration is available, however.

Notice that condition (G) is not satisfied in the present setting (thus $(H)$ does not hold neither). Furthermore, both $F$ and $\Lambda$ are continuous processes, but $F$ is not an increasing process,[9] and clearly $\Gamma \ne \Lambda$. To conclude, if (G) fails to hold, the continuity of $\Gamma$ and $\Lambda$ is not sufficient for the equality $\Gamma = \Lambda$ to be satisfied. Let us finally observe that **F**-intensity $\Lambda$ is also the $\hat{\mathbf{F}}$-intensity of $\tau$, where $\hat{\mathbf{F}}$ stands for the filtration generated by the process $|W_t|$ (of course, $\hat{\mathbf{F}}$ is a strict subfiltration of **F**). We have $\Gamma_t = P(g \le t \,|\, \mathcal{F}_t) = P(g \le t \,|\, \hat{\mathcal{F}}_t)$ for every $t$.

- **Defaultable claims**

We present here the computation of $E(H(W_T)\mathbb{1}_{\{T<g\}} \,|\, \mathcal{G}_t)$, for $t < T < 1$. We know that

$$P(g > t \,|\, \mathcal{F}_t) = 1 - \Phi\left( \frac{|W_t|}{\sqrt{1-t}} \right).$$

In order to evaluate $E(H(W_T)\mathbb{1}_{\{T<g\}} \,|\, \mathcal{F}_t)$, we first condition with respect to $\mathcal{F}_T$:

$$E(H(W_T)\mathbb{1}_{\{T<g\}} \,|\, \mathcal{F}_t) = E(H(W_T) \,|\, \mathcal{F}_t) - E\Big(H(W_T)\Phi\Big(\frac{|W_T|}{\sqrt{1-T}}\Big) \,\Big|\, \mathcal{F}_t\Big).$$

The computation can be done using the Markov property: $E(H(W_T) \,|\, \mathcal{F}_t) = \widehat{H}(W_t)$, where $\widehat{H}(a) = E(H(W_{T-t} + a))$ and

$$E\Big(H(W_T)\Phi\Big(\frac{|W_T|}{\sqrt{1-T}}\Big) \,\Big|\, \mathcal{F}_t\Big) = \widehat{H\Phi}(W_t),$$

[9] Thus formula (59) rather than (54) should be used to evaluate $\Lambda$.

where in turn

$$\widehat{H\Phi}(a) = E\Big(H(W_{T-t}+a)\Phi\Big(\frac{|\,W_{T-t}+a\,|}{\sqrt{1-T}}\Big)\Big),$$

or more explicitly,

$$\widehat{H\Phi}(a) = \frac{1}{\sqrt{2\pi(T-t)}}\int_{I\!R} H(u)\Phi\Big(\frac{|\,u\,|}{\sqrt{1-T}}\Big)\exp\Big(-\frac{(u-a)^2}{2(t-t)}\Big)\,du.$$

The computation for the rebate part, in the particular case $h_s = h(V_s)$ follows from

$$E(\int_t^T h_s d\widetilde{F}_s|\mathcal{F}_t) = h(a)E(\widetilde{F}_T - \widetilde{F}_t|\mathcal{F}_t) = h(a)[E(e^{\Gamma_T}|\mathcal{F}_t) - e^{\Gamma_t}]$$

which can be done with Markov property.

- **Generalisation**

These results can be extended to the last time before 1 where the Brownian motion reaches the level $\alpha$. Let $g^\alpha = \sup\{t \le 1 \,:\, W_t = \alpha\}$, where $\sup(\emptyset) = 1$. In the filtration $\mathbf{H}$, the intensity function of $g^\alpha$ is given through the probability law of $g^\alpha$, as given in Yor [58] (see formula (3.b) therein)

$$P(g^\alpha \in du) = \exp\Big(-\frac{\alpha^2}{2(1-u)}\Big)\frac{du}{\pi\sqrt{u(1-u)}}.$$

Note that the right-hand side is a sub-probability, and that the missing mass is

$$P(g^\alpha = 1) = P(\tau_\alpha \ge 1) = P(\,|\,G\,| \le \alpha)\,.$$

where $G$ is the standard Gaussion variable. The computation in the enlarged filtration follows from the remark that

$$P(g^\alpha < t\,|\,\mathcal{F}_t) = \Phi\left(\frac{|\,\alpha - W_t\,|}{\sqrt{1-t}}\right).$$

The dual predictable projection of $N_t = 1\!\!1_{\{g^\alpha \le t\}}$ is

$$A_t = \sqrt{\frac{2}{\pi}}\int_0^t \frac{dL_s^\alpha}{\sqrt{1-s}},$$

where $L^\alpha$ is the local time of $W$ at the level $\alpha$.

### 6.4.2 Brownian Motion with Drift

We return to the general case (74) with constant coefficients: $dV_t = \mu\, dt + \sigma\, dW_t$.

**• Computation of the intensity function**

In this section, we compute the intensity of the default time in the filtration $\mathbf{H}$. In order to simplify the proof, we set $\theta = 1$, and we write $g^a(V)$ for $g_1^a(V)$. Of course, we need only to know the probability law $P(g^a(V) < t)$. The default time can be written in terms of Brownian motion as follows

$$\begin{aligned} g^a(V) &= \sup\{t \le 1 \,:\, \mu t + \sigma W_t = a - v\} \\ &= \sup\{t \le 1 \,:\, \nu t + W_t = \alpha\} \\ &= \sup\{t \le 1 \,:\, \tilde{W}_t = \alpha\} = g^\alpha(\tilde{W}) \end{aligned}$$

where $\nu = \mu/\sigma$ and $\alpha = (a - v)/\sigma < 0$. Using Girsanov's theorem, we obtain

$$P(g^a(V) \le t) = E\Big(1\!\!1_{\{g^\alpha \le t\}} \exp\big(\nu W_1 - \frac{\nu^2}{2})\Big) \tag{77}$$

where

$$g^\alpha = g_1^\alpha(W) = \sup\{t \le 1 \,:\, W_t = \alpha\}.$$

Then

$$P(g^a(V) \le t) = \exp\Big(\nu\alpha - \frac{\nu^2}{2}\Big)\, E\Big(1\!\!1_{\{g^\alpha(W) < t\}} \exp\big(\nu \epsilon m_1 \sqrt{1 - g^\alpha}\big)\Big)$$

where $\epsilon$ is a Bernoulli random variable, $m_1$ is the value at time 1 of the Brownian meander, with the probability law

$$P(m_1 \in dx) = x \exp(-\frac{x^2}{2})\, 1\!\!1_{\{x \ge 0\}}\, dx.$$

Furthermore, the random variables $g^\alpha, \epsilon$, and $m_1$ are mutually independent. Therefore,

$$\begin{aligned} P(g^a(V) \le t) &= \exp\Big(\nu\alpha \\ &- \frac{\nu^2}{2}\Big) \int_0^t \frac{1}{\pi\sqrt{u(1-u)}} \exp\Big(-\frac{\alpha^2}{2(1-u)}\Big) \Upsilon(\nu, u)\, du \\ &\stackrel{def}{=} \Psi(v, a, t), \end{aligned} \tag{78}$$

where

$$\Upsilon(\nu,u)=\frac{1}{2}\Big(\int_0^\infty e^{\nu x\sqrt{1-u}}xe^{-x^2/2}\,dx+\int_0^\infty e^{-\nu x\sqrt{1-u}}xe^{-x^2/2}\,dx\Big),$$

that is,

$$\Upsilon(\nu,u)=\int_0^\infty \cosh(-\nu x\sqrt{1-u})xe^{-x^2/2}\,dx.$$

Of course, for $\nu=0$ we obtain the same result as in the previous paragraph.

**Proposition 6.1** *When the agent has only the information of default time, the value of a defaultable zero-coupon bond is* $\exp(-\int_t^T \lambda_s\,ds)$, *where* ($\Psi$ *is defined by* (78))

$$\lambda_t=\frac{\Psi_t'(v,a,t)}{1-\Psi(v,a,t)}\,.$$

- **Computation of the F-hazard process**

Following the above computations, we obtain

$$P(g^a(V)\le t\,|\,\mathcal{F}_t)=D_t^{-1}\,E(D_1 1\!\!1_{\{g^\alpha(W)\le t\}}\,|\,\mathcal{F}_t),$$

where $D_t=\exp\big(\nu W_t-\dfrac{t\nu^2}{2}\big)$. Therefore, from the equality

$$\{g^\alpha(W)\le t\}=\{\tau_\alpha(W)\le t\}\cap\{d_t^\alpha>1\}$$

we obtain

$$\begin{aligned}&E(D_1 1\!\!1_{\{g^\alpha\le t\}}\,|\,\mathcal{F}_t)\\&\quad=\exp\big(\nu W_t-\frac{\nu^2}{2}\big)\,1\!\!1_{\{\tau_\alpha(W)\le t\}}E\big(\exp[\nu(W_1-W_t)]1\!\!1_{\{d_t^\alpha(W)>1\}}\,|\,\mathcal{F}_t\big)\end{aligned}$$

Using the independence properties of the Brownian motion and equality (75), we get

$$\begin{aligned}&E\big(\exp[\nu(W_1-W_t)]1\!\!1_{\{d_t^\alpha(W)>1\}}\,|\,\mathcal{F}_t\big)\\&\quad=E\big(\exp[\nu\widehat{W}_{1-t}]1\!\!1_{\{\widehat{\tau}_{-W_t}>1-t\}}\,|\,\mathcal{F}_t\big)=\Theta(\alpha-W_t,1-t)\end{aligned}$$

where

$$\Theta(w,s) = E\big(e^{\nu W_s} 1\!\!1_{\{\tau_w \geq s\}}\big) = e^{s\nu^2/2} - E\big(e^{\nu W_s} 1\!\!1_{\{\tau_w < s\}}\big).$$

• **Computation of** $\Theta$

By conditioning with respect to $\mathcal{F}_{\tau_w}$, we obtain

$$\begin{aligned}
&E\big(e^{\nu W_s} 1\!\!1_{\{\tau_w < s\}}\big) \\
&\quad = e^{\nu w} E\Big(1\!\!1_{\{\tau_w < s\}} e^{\frac{\nu^2}{2}(s-\tau_w)} E\big(e^{\nu(W_s - W_{\tau_w}) - \frac{\nu^2}{2}(s-\tau_w)} \,|\, \mathcal{F}_{\tau_w}\big)\Big) \\
&\quad = e^{\nu w} E\Big(1\!\!1_{\{\tau_w < s\}} e^{\frac{\nu^2}{2}(s-\tau_w)}\Big) = e^{\nu w + \frac{s\nu^2}{2}} H(\nu, \,|\, w \,|\, , s)
\end{aligned}$$

where $H$ is defined in (1). Therefore,

$$\begin{aligned}
P(g^a(V) \leq t \,|\, \mathcal{F}_t) &= 1\!\!1_{\{\tau_\alpha(W) \leq t\}} \exp\Big(\frac{(u-1)\nu^2}{2}\Big) \Theta(\alpha - W_t, 1-t) \\
&= 1\!\!1_{\{\tau_\alpha(W) \leq t\}} \Big(1 - e^{\nu(\alpha - W_t)} H(\nu, \,|\, \alpha - W_t \,|\, , 1-t)\Big).
\end{aligned}$$

**Lemma 6.3** *We have*

$$P(g^a(V) > t \,|\, \mathcal{F}_t) = 1\!\!1_{\{\tau_\alpha(W) \leq t\}} e^{\nu(\alpha - W_t)} H(\nu, \,|\, \alpha - W_t \,|\, , 1-t).$$

• **Value of a defaultable zero-coupon bond**

We can express the above result in terms of the value of the firm, using that the Brownian motion that we are using is equal to $(V_t - v)/\sigma$. We have

$$P(g^a(V) \leq t \,|\, \mathcal{F}_t) = 1\!\!1_{\{\tau_a(V) \leq t\}} \Big(1 - e^{-\nu(V_t - a)/\sigma} H(\nu, \frac{1}{\sigma} \,|\, V_t - a \,|\, , 1-t)\Big).$$

The computation of $P(g^a(V) > T \,|\, \mathcal{F}_t) = E(Z_T \,|\, \mathcal{F}_t)$ can be done as in he previous case, with the help of Markov property.

• **Decomposition of** $P(g^a(V) \leq t \,|\, \mathcal{F}_t)$

Using the decomposition $P(g^a(V) \leq t \,|\, \mathcal{F}_t) = Y_t 1\!\!1_{\{\tau_\alpha(W) \leq t\}}$, where

$$Y_t = 1 - e^{\nu(\alpha - W_t)} H(\nu, \,|\, \alpha - W_t \,|\, , 1-t),$$

we are able to determine the **F**-martingale hazard process of $g^a(V)$. Using Itô's lemma, we obtain the decomposition of $Y_t$ as a semimartingale $Y_t = M_t + K_t$. More precisely $dM_t = m_t dW_t$ and $dK_t = k_t dt + \kappa_t dL_t$, where

$$\begin{aligned} m_t = &\ e^{\nu(\alpha - W_t)}(\nu H - \operatorname{sgn}(\alpha - W_t)\, H'_x)(\nu, \,|\, \alpha - W_t \,|\, , 1-t), \\ k_t = &\ e^{\nu(\alpha - W_t)}(H'_u + \nu H'_x \operatorname{sgn}(\alpha - W_t) \\ &\quad - (1/2)(H''_{xx} + \nu^2 H))(\nu, \,|\, \alpha - W_t \,|\, , 1-t), \\ \kappa_t = &\ -e^{\nu(\alpha - W_t)}\, H'_x(\nu, \,|\, \alpha - W_t \,|\, , 1-t). \end{aligned}$$

Therefore, the **F**-compensator of $g^a(V)$ is $1\!\!1_{\{\tau_\alpha(W) \le t\}} \dfrac{dK_t}{1 - Y_t}$.

## 6.5 Absolutely Continuous Intensity

We give here a purely mathematical example,[10] in which the default time is $\mathcal{F}_\infty$-measurable random variable, and the **F**-martingale hazard process is absolutely continuous with respect to Lebesgue's measure.

Let $W$ be a Brownian motion and let $\tau = \sup\{t \le 1 \,:\, W_t = W_1/2\}$. Then,

$$\{\tau \le t\} = \{\inf_{t \le s \le 1} 2W_s \ge W_1 \ge 0\} \cup \{\sup_{t \le s \le 1} 2W_s \le W_1 \le 0\}.$$

The quantity $P(\tau \le t, W_1 \ge 0 \,|\, \mathcal{F}_t)$ can be evaluated using the equalities

$$\begin{aligned} \{\inf_{t \le s \le 1} W_s \ge \frac{W_1}{2} \ge 0\} &= \{\inf_{t \le s \le 1} (W_s - W_t) \ge \frac{W_1}{2} - W_t \ge -W_t\} \\ &= \{\inf_{0 \le u \le 1-t} (\widetilde{W}_u) \ge \frac{\widetilde{W}_{1-t}}{2} - \frac{W_t}{2} \ge -W_t\}, \end{aligned}$$

where $\widetilde{W}$ is a Brownian motion independent of $\mathcal{F}_t$. More precisely, $P(\tau \le t, W_1 \ge 0 \,|\, \mathcal{F}_t) = \Psi(t, W_t)$, where

$$\begin{aligned} \Psi(t,x) &= P(\inf_{0 \le u \le 1-t} \widetilde{W}_u \ge \frac{\widetilde{W}_{1-t}}{2} - \frac{1}{2}x \ge -x) \\ &= P(\sup_{0 \le u \le 1-t} W_u \le \frac{W_{1-t}}{2} + \frac{1}{2}x \le x) \\ &= P(2S_{1-t} - W_{1-t} \le x, W_{1-t} \le x). \end{aligned}$$

[10] We are indebted to Michel Emery for this example.

Then $Z_t = P(\tau > t \,|\, \mathcal{F}_t) = 2\Psi(t, W_t)$, and the $\mathbf{F}$-martingale hazard process of the first jump process $N_t = \mathbb{1}_{\{\tau \leq t\}}$ satisfies

$$d\widetilde{F}_t = \Big(2\,\frac{\partial \Psi}{\partial t} + \frac{\partial^2 \Psi}{\partial x^2}\Big)(t, W_t)\,dt.$$

## 6.6 Information

Suppose that the value of the firm is an asset, or that there is an asset such that the market is $\mathcal{F}_T$-complete, i.e. any $\mathcal{F}_T$-measurable r.v. is hedgeable. Suppose that, as in the last example, the default time is $\mathcal{F}_T$-measurable. In this case, if the information of an agent is the filtration $\mathcal{G}_t$, this agent will be an inside trader. In an obvious way, if the agent observes the last passage time at a fixed level, as soon as this time is revealed, he will know that the price will stay below (or above) this level. Investigating this kind of information is work in progress. If there is no inside trader, the price of a defaultable claim will be hedgeable.

## 6.7 Representation Theorem

In the present setting, under some extra hypothesis on $\tau$ (i.e. $\tau$ is *honest*), a representation theorem can be established (see the paper by Azéma, Jeulin, Knight and Yor [2]). The basic martingales are here the Brownian motion in the enlarged filtration, the discontinuous martingale $M$, and the martingales $v\mathbb{1}_{\{\tau \leq t\}}$, where $v \in \mathcal{F}_\tau^+$, $E(v \,|\, \mathcal{F}_\tau) = 0$.

Here $\mathcal{F}_\tau^+$ stands for $\sigma\{h_\tau : h \text{ any } \mathbf{F}\text{-progressively measurable process}\}$, and $\mathcal{F}_\tau$ stands for $\sigma\{h_\tau : h \text{ any } \mathbf{F}\text{-optional process}\}$. In a more explicit form, any $\mathbf{G}$-martingale $Z$ can be written as (cf. Theorem 3 in [2])

$$Z_t = \widehat{Z}_t^{(1)} + \widehat{Z}_t^{(2)} + \widehat{Z}_t^{(3)},$$

where the $\widehat{Z}^{(i)}$ are orthogonal $\mathbf{G}$-martingales of the form

$$\widehat{Z}_t^{(1)} = \int_0^t \Psi_s^{(1)} d\widehat{B}_t,\; \widehat{Z}_t^{(2)} = \int_0^t \Psi_s^{(2)} dM_t,\; \widehat{Z}_t^{(3)} = v\mathbb{1}_{\tau \leq t}.$$

It seems that the space $v\mathbb{1}_{\{\tau \leq t\}}$, where $v \in \mathcal{F}_\tau^+$, $E(v \,|\, \mathcal{F}_L) = 0$ is a one-dimensional space. This result seems difficult to extend to the general case where $\tau$ is not $\mathcal{F}_\infty$-measurable.

# 7 Standard Constructions of Random Times

We shall describe a standard construction of a random time $\tau$, in which this property is valid. Subsequently, we shall examine the properties of the minimum of several random times.

## 7.1 Random Time with a Given Hazard Process

In this section, we shall examine the commonly used construction of a random time for a given hazard process $\Psi$. The $\mathbf{F}$-adapted continuous hazard process $\Psi$ can be equally well considered as the $\mathbf{F}$-hazard process $\Gamma$, or the $\mathbf{F}$-martingale hazard process $\Lambda$. Indeed, in the standard construction of $\tau$ the following properties hold:
(i) $\Psi$ coincides with the $\mathbf{F}$-hazard process $\Gamma$ of a random time $\tau$,
(ii) $\Psi$ is also the $\mathbf{F}$-martingale hazard process of a random time $\tau$, and
(iii) $\Psi$ is $\mathbf{G}$-martingale hazard process of a $\mathbf{G}$-stopping time $\tau$.
It should be noticed that $\tau$ constructed below is merely a random time (but not a stopping time) with respect to the filtration $\mathbf{F}$, and it is a totally inaccessible stopping time with respect to the enlarged filtration $\mathbf{G}$.

Let $\Psi$ be a $\mathbf{F}$-adapted, continuous, increasing process given on a probability space $(\widetilde{\Omega}, (\mathcal{F}_t)_{t \in I\!R_+}, \widetilde{P})$ such that $\Psi_\infty = +\infty$. For instance, $\Psi$ can be given by the formula

$$\Psi_t = \int_0^t \psi_u \, du, \quad \forall \, t \in I\!R_+, \tag{79}$$

where $\psi$ is a nonnegative $\mathbf{F}$-predictable process. Our goal is to construct a random time $\tau$, on an enlarged probability space $(\Omega, \mathcal{G}, P)$, in such a way that $\Psi$ is a $\mathbf{F}$-(martingale) hazard process of $\tau$. To this end, we assume that $\xi$ is a random variable on some probability space[11] $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, with the uniform probability law on $[0,1]$. We take $\Omega = \widetilde{\Omega} \times \hat{\Omega}$, $\mathcal{G} = \mathcal{F}_\infty \otimes \hat{\mathcal{F}}$ and $P = \widetilde{P} \otimes \hat{P}$. We introduce the random time $\tau$ through the formula

$$\tau = \inf \{ \, t \in I\!R_+ \, : \, e^{-\Psi_t} \leq \xi \, \} = \inf \{ \, t \in I\!R_+ \, : \, \Psi_t \geq -\ln \xi \, \}. \tag{80}$$

[11] Of course, it is enough to assume that we may define on $(\Omega, \mathcal{G}, P)$ a random variable $\xi$ which is uniformly distributed on $[0,1]$, and is independent of the process $\Psi$ (we then set $\hat{\mathcal{F}} = \sigma(\xi)$).

Also, we set $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for every $t$,. We shall now check that properties (i)-(iii) also hold.

PROOF OF (i): Let us first check that (i) holds. To this end, we shall find the process $F_t = P(\tau \leq t \,|\, \mathcal{F}_t)$. Since clearly $\{\tau > t\} = \{e^{-\Psi_t} > \xi\}$, we get $P(\tau > t \,|\, \mathcal{F}_\infty) = e^{-\Psi_t}$. Consequently,

$$1 - F_t = P(\tau > t \,|\, \mathcal{F}_t) = E\big(P(\tau > t \,|\, \mathcal{F}_\infty) \,|\, \mathcal{F}_t\big) = e^{-\Psi_t},$$

and thus $F$ is a $\mathbf{F}$-adapted continuous increasing process. Notice that

$$F_t = 1 - e^{-\Psi_t} = P(\tau \leq t \,|\, \mathcal{F}_\infty) = P(\tau \leq t \,|\, \mathcal{F}_t). \tag{81}$$

We conclude that $\Psi$ coincides with the $\mathbf{F}$-hazard process $\Gamma$.

PROOF OF (ii): The next step is to check that $\Psi$ is the $\mathbf{F}$-martingale hazard process $\Lambda$. This can be done either directly, or through equality $\Lambda = \Gamma$. Since $\Psi$ is a continuous process, to show that $\Lambda = \Gamma$, it is enough to check that condition (H$'$) (or equivalently, condition (H)) holds, and to apply Corollary 5.3.

Let us first check that (H$'$) is valid (cf (53)). We fix $t$ and we consider an arbitrary $u \leq t$. Since for any $u \in I\!R_+$

$$P(\tau \leq u \,|\, \mathcal{F}_\infty) = 1 - e^{-\Psi_u}, \tag{82}$$

we indeed obtain for $u \leq t$

$$P(\tau \leq u \,|\, \mathcal{F}_t) = E\big(P(\tau \leq u \,|\, \mathcal{F}_\infty) \,\big|\, \mathcal{F}_t\big) = 1 - e^{-\Psi_u} = P(\tau \leq u \,|\, \mathcal{F}_\infty).$$

Alternatively, we may also check directly that (H) holds. Since $\{\tau \leq s\} = \{\Psi_s \geq -\ln \xi\} \in \hat{\mathcal{F}} \vee \mathcal{F}_s$, it is clear that $\mathcal{F}_t \subset \mathcal{H}_t \vee \mathcal{F}_t \subset \hat{\mathcal{F}} \vee \mathcal{F}_t$. Therefore, for any $\mathcal{F}_\infty$-measurable bounded random variable $\xi$, we have

$$E(\xi \,|\, \mathcal{H}_t \vee \mathcal{F}_t) = E(\xi \,|\, \hat{\mathcal{F}} \vee \mathcal{F}_t) = E(\xi \,|\, \mathcal{F}_t), \tag{83}$$

where the second equality is a consequence of the independence of $\hat{\mathcal{F}}$ and $\mathcal{F}_\infty$. This shows that (H) holds.

We conclude that the $\mathbf{F}$-martingale hazard process $\Lambda$ of $\tau$ coincides with $\Gamma$. To be more specific, we have $\Psi_t = \Lambda_t = \Gamma_t = -\ln(1 - F_t)$. Furthermore, any $\mathbf{F}$-martingale is also a $\mathbf{G}$-martingale.

PROOF OF (iii). Let us now check directly that $\Psi$ is a $\mathbf{F}$-martingale hazard process of a random time $\tau$. Since $\Psi$ is a $\mathbf{F}$-predictable process (and thus a **G**-predictable process), we shall simultaneously show that $\Psi$ is also the **G**-martingale hazard process of a **G**-stopping time $\tau$. We need to verify that the process $N_t - \Psi_{t\wedge\tau}$ follows a **G**-martingale. First, by virtue of Lemma 5.1 we have for $t \le s$

$$\begin{aligned} E(N_s - N_t \,|\, \mathcal{G}_t) &= E(\mathbb{1}_{\{t<\tau\le s\}} \,|\, \mathcal{G}_t) \\ &= \mathbb{1}_{\{\tau>t\}}\, E(\mathbb{1}_{\{t<\tau\le s\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{P(t<\tau\le s \,|\, \mathcal{F}_t)}{P(\tau>t \,|\, \mathcal{F}_t)}. \end{aligned}$$

Using (81), we get $P(t<\tau\le s \,|\, \mathcal{F}_t) = E(F_s \,|\, \mathcal{F}_t) - F_t$. Therefore

$$E(N_s - N_t \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{E(F_s \,|\, \mathcal{F}_t) - F_t}{1-F_t}. \tag{84}$$

On the other hand, if we set $Y = \Psi_{s\wedge\tau} - \Psi_{t\wedge\tau}$, then in view of (i) we get (cf. (55))

$$Y = \mathbb{1}_{\{\tau>t\}} Y = \ln\Big(\frac{1-F_{s\wedge\tau}}{1-F_{t\wedge\tau}}\Big) = \int_{]t,s\wedge\tau]} \frac{dF_u}{1-F_u}.$$

Using again (42), we obtain (for the last equality in the formula below, see (56))

$$\begin{aligned} E\big(Y \,|\, \mathcal{G}_t\big) &= \mathbb{1}_{\{\tau>t\}} \frac{E(Y \,|\, \mathcal{F}_t)}{P(\tau>t \,|\, \mathcal{F}_t)} \\ &= \mathbb{1}_{\{\tau>t\}} \frac{E\big(\int_{]t,s\wedge\tau]} (1-F_u)^{-1} dF_u \,|\, \mathcal{F}_t\big)}{1-F_t} = \mathbb{1}_{\{\tau>t\}} \frac{E(F_s \,|\, \mathcal{F}_t) - F_t}{1-F_t}. \end{aligned}$$

We conclude that the process $N_t - \Psi_{t\wedge\tau}$ is indeed a **G**-martingale.

Notice that the role played by the 'hazard process' $\Psi$ in (i) and (iii) is slightly different. If we consider $\Psi$ as a **F**-hazard process of $\tau$, then using Corollary 4.2 we deduce that for any $\mathcal{F}_s$-measurable random variable $Y$

$$E(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} E\big(Y e^{\Gamma_t - \Gamma_s} \,|\, \mathcal{F}_t\big). \tag{85}$$

On the other hand, if $\Psi$ is considered as the **G**-martingale hazard process then, for any $\mathcal{G}_s$-measurable random variable $Y$ such that the associated process $V$ is continuous at $\tau$ we obtain

$$E(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} E\big(Y e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{G}_t\big). \tag{86}$$

If $Y$ is actually $\mathcal{F}_s$-measurable then we have (see (83))

$$E\big(Ye^{\Lambda_t-\Lambda_s}\,|\,\mathcal{G}_t\big) = E\big(Ye^{\Lambda_t-\Lambda_s}\,|\,\mathcal{F}_t\vee\mathcal{H}_t\big) = E\big(Ye^{\Lambda_t-\Lambda_s}\,|\,\mathcal{F}_t\big)$$

so that the associated process $V$ is necessarily continuous at $\tau$, and formulae (85) and (86) coincide.

**Remark 7.1** Assume that $\Psi$ satisfies (79). Then (84) can be rewritten as follows

$$P(\tau > t \leq s\,|\,\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, E\big(1 - e^{-\int_t^s \psi_u\,du}\,|\,\mathcal{F}_t\big). \tag{87}$$

Using (82), we find that the cumulative distribution function of a random time $\tau$ under $P$ equals

$$P(\tau \leq t) = 1 - E_{\hat{P}}\Big(e^{-\int_0^t \psi_u\,du}\Big) = 1 - e^{-\int_0^t \gamma^0(u)\,du},$$

where we write $\gamma^0$ to denote the unique intensity function (i.e., $\mathbf{F}^0$-intensity) of $\tau$.

## 7.2 Ordered Random Times

Consider now two $\mathbf{F}$-adapted continuous processes, $\Psi^1$ and $\Psi^2$, which satisfy $\Psi^2_t > \Psi^1_t$ for every $t \in \mathbb{R}_+$. For $i = 1,2$ we set

$$\tau_i = \inf\{\, t \in \mathbb{R}_+ \,:\, e^{-\Psi^i_t} \leq \xi\,\} = \inf\{\, t \in \mathbb{R}_+ \,:\, \Psi^i_t \geq -\ln \xi\,\}. \tag{88}$$

so that obviously $\tau_1 < \tau_2$ with probability 1.

We shall write $\mathbf{G}^i = \mathbf{H}^i \vee \mathbf{F}$, for $i = 1,2$, and $\mathbf{G} = \mathbf{H}^1 \vee \mathbf{H}^2 \vee \mathbf{F}$. An individual analysis of each random time $\tau_i$ in its respective filtration $\mathbf{G}^i$ can be done along the same lines as in the previous section. It is thus clear that the process $\Psi^i$ represents: the $\mathbf{F}$-hazard process $\Gamma^i$ of $\tau_i$, the $\mathbf{F}$-martingale hazard process $\Lambda^i$ of $\tau_i$, and finally the $\mathbf{G}^i$-martingale hazard process of $\tau_i$ when $\tau_i$ is considered as a $\mathbf{G}^i$-stopping time. Therefore, we shall focus on the study of hazard processes of $\tau_i$'s with respect to enlarged filtrations. We find it convenient to introduce the following auxiliary notation:[12] $\mathbf{F}^i = \mathbf{H}^i \vee \mathbf{F}$, so that $\mathbf{G} = \mathbf{H}^1 \vee \mathbf{F}^2$ and $\mathbf{G} = \mathbf{H}^2 \vee \mathbf{F}^1$.

[12] Though $\mathbf{F}^i = \mathbf{G}^i$, this double notation seems to be useful anyway.

Let us start by an analysis of $\tau_1$. We search for the $\mathbf{F}^2$-hazard process $\widetilde{\Gamma}^1$ of $\tau_1$, as well as for the $\mathbf{F}^2$-martingale hazard process $\widetilde{\Lambda}^1$ of $\tau_1$. We shall first check that $\widetilde{\Gamma}^1 \neq \Gamma^1$. Indeed, for every $t \in \mathbb{R}_+$ we have

$$e^{-\widetilde{\Gamma}^1_t} = P(\tau_1 > t \,|\, \mathcal{F}^2_t) = P(\tau_1 > t \,|\, \mathcal{F}_t \vee \mathcal{H}^2_t)$$

and

$$e^{-\Gamma^1_t} = P(\tau_1 > t \,|\, \mathcal{F}_t) = e^{-\Psi^1_t}.$$

$\widetilde{\Gamma}^1 \neq \Gamma^1$ would thus imply

$$P(\tau_1 > t \,|\, \mathcal{F}_t \vee \mathcal{H}^2_t) = P(\tau_1 > t \,|\, \mathcal{F}_t), \quad \forall\, t \in \mathbb{R}_+.$$

The last equality does not hold, however. In effect, the inequality $\tau_2 \leq t$ implies $\tau_1 \leq t$, therefore on the set $\{\tau_2 \leq t\}$, which clearly belongs to $\mathcal{H}^2_t$, we have $P(\tau_1 > t \,|\, \mathcal{F}_t \vee \mathcal{H}^2_t) = 0$. We conclude that the $\mathbf{F}^2$-hazard process $\widetilde{\Gamma}^1$ is well defined only strictly before $\tau_2$. Furthermore, it can be checked that the stopped process $\widetilde{\Gamma}^1_{t\wedge\tau_1}$ coincides with $\Gamma^1_{t\wedge\tau_1}$.

On the other hand, since $\mathbf{G} = \mathbf{G}^1 \vee \mathbf{H}^2$, it is clear that the process $N^1_t - \Psi^1_{t\wedge\tau_1}$ is not only a $\mathbf{G}^1$-martingale, but also a $\mathbf{G}$-martingale. This shows that $\Psi^1$ coincides with the $\mathbf{F}^2$-martingale hazard process $\widetilde{\Lambda}^1$ of $\tau_1$, Finally, $\Psi^1$ plays also the role the $\mathbf{G}$-martingale hazard process $\hat{\Lambda}^1$ of $\tau_1$.

As one might easily guess, the properties of $\tau_2$ with respect to the filtration $\mathbf{F}^1$ are quite different. We have

$$e^{-\widetilde{\Gamma}^2_t} = P(\tau_2 > t \,|\, \mathcal{F}^1_t) = P(\tau_2 > t \,|\, \mathcal{F}_t \vee \mathcal{H}^1_t).$$

We claim that $\widetilde{\Gamma}^2 \neq \Gamma^2$, that is, the equality

$$P(\tau_2 > t \,|\, \mathcal{F}_t \vee \mathcal{H}^1_t) = P(\tau_2 > t \,|\, \mathcal{F}_t)$$

is not valid, in general. Indeed, the inequality $\tau_1 > t$ implies $\tau_2 > t$, and thus on set $\{\tau_1 > t\}$, which belongs to $\mathcal{H}^1_t$, we have $P(\tau_2 > t \,|\, \mathcal{F}_t \vee \mathcal{H}^1_t) = 1$. We deduce easily that the process $\widetilde{\Gamma}^2$ is not well defined after time $\tau_1$.

Furthermore, the process $N^2_t - \Psi^2_{t\wedge\tau_2}$ is a $\mathbf{G}^2$-martingale; it does not follow a $\mathbf{G}$-martingale, however (otherwise, equality $\widetilde{\Gamma}^2 = \Gamma^2 = \Psi^2$ would be true up to time $\tau_2$, but this clearly does not hold). It seems rather difficult to evaluate exactly the $\mathbf{F}^1$-martingale hazard process $\widetilde{\Lambda}^2$ of $\tau_2$, it is reasonable to expect that it is discontinuous at $\tau_1$.

Let us finally notice that $\tau_1$ is a totally inaccessible stopping time not only with respect to $\mathbf{G}^1$, but also with respect to the joint filtration $\mathbf{G}$. On the other hand, $\tau_2$ is a totally inaccessible stopping time with respect to $\mathbf{G}^1$, but it is a predictable stopping time with respect to $\mathbf{G}$. Indeed, we may find an announcing sequence of $\mathbf{G}$-stopping times

$$\tau_2^n = \inf\{\, t \geq \tau_1 \,:\, \Psi_t^2 \geq -\ln(\xi) - \tfrac{1}{n}\,\}.$$

Therefore the $\mathbf{G}$-martingale hazard process $\hat{\Lambda}^2$ of $\tau_2$ coincides with the $\mathbf{G}$-predictable process $N_t^2 = \mathbb{1}_{\{\tau_2 \leq t\}}$.

Let us set $\tau = \tau_1 \wedge \tau_2$. In the present setup, we have $\tau = \tau_1$, and thus the $\mathbf{G}$-martingale hazard process $\hat{\Lambda}$ of $\tau$ is equal to $\Psi^1$. It is also equal to the sum of $\mathbf{G}$-martingale hazard processes $\hat{\Lambda}^i$ of $\tau_i$, $i = 1, 2$, stopped at $\tau$. More precisely, we have

$$\hat{\Lambda}_{t\wedge\tau} = \Psi^1_{t\wedge\tau} = \Psi^1_{t\wedge\tau} + N^2_{t\wedge\tau} = \hat{\Lambda}^1_{t\wedge\tau} + \hat{\Lambda}^2_{t\wedge\tau}.$$

We shall see in the next section that this property is universal (though not necessarily very useful).

## 7.3 Properties of the Minimum of Several Random Times

In this section, we shall examine the following problem: given a finite family of random times $\tau_i$, $i = 1, \ldots, n$, and the associated hazard processes, find the hazard process of the random time $\tau = \min(\tau_1, \ldots, \tau_n)$. Of course, the problem above cannot be solved in such a generality, that is, without the knowledge of the joint law of $(\tau_1, \ldots, \tau_n)$. Indeed, as we shall see in what follows the solution depends heavily on specific assumptions on random times and the choice of filtrations.

### 7.3.1 Hazard Function of the Minimum of Several Random Times

Let us first consider a simple result, in which we focus on the calculation of the hazard function of the minimum of several independent random times.

**Lemma 7.1** *Let $\tau_i$, $i = 1, \ldots, n$, be $n$ random times defined on a common probability space $(\Omega, \mathcal{G}, P)$. Assume that $\tau_i$ admits the hazard function $\Gamma^i$. If $\tau_i$, $i = 1, \ldots, n$, are mutually independent random variables, then the hazard function $\Gamma$ of $\tau$ is the sum of hazard functions $\Gamma^i$, $i = 1, \ldots, n$.*

PROOF: For any $t \in I\!R_+$ we have

$$\begin{aligned} e^{-\Gamma(t)} &= 1 - F(t) = P(\tau > t) = P(\min(\tau_1, \ldots, \tau_n) > t) \\ &= \prod_{i=1}^{n} P(\tau_i > t) = \prod_{i=1}^{n} (1 - F_i(t)) \\ &= \prod_{i=1}^{n} e^{-\Gamma^i(t)} = e^{-\sum_{i=1}^{n} \Gamma^i(t)}. \qquad \square \end{aligned}$$

$\square$

Let us now focus on the case of continuous distribution functions $F_i$, $i = 1, \ldots, n$. In this case, we get also $\Lambda(t) = \sum_{i=1}^{n} \Lambda^i(t)$. In particular, if $\tau_i$ admit intensities $\gamma_t^i = \lambda_t^i = f_i(t)(1 - F_i(t))^{-1}$, then the process (as usual, $N_t = 1\!\!1_{\{\tau \le t\}}$)

$$N_t - \int_0^{t \wedge \tau} \sum_{i=1}^{n} \gamma^i(u)\, du = N_t - \int_0^{t \wedge \tau} \sum_{i=1}^{n} \lambda^i(u)\, du$$

is a $\mathbf{H}$-martingale (notice that $\mathbf{H} = \mathbf{H}^1 \vee \cdots \vee \mathbf{H}^n$).

Conversely, if the hazard function of $\tau$ satisfies $\Lambda = \Gamma = \sum_{i=1}^{n} \Gamma^i = \sum_{i=1}^{n} \Lambda^i$, then we obtain

$$P(\tau_1 > t, \ldots, \tau_n > t) = \prod_{i=1}^{n} P(\tau_i > t), \quad \forall\, t \in I\!R_+.$$

### 7.3.2 Martingale Hazard Process of the Minimum of Several Stopping Times

In this section, we shall assume that $\tau_1, \ldots, \tau_n$ are stopping times with respect to some filtration $\mathbf{F}$. Formally, $\mathbf{H}^i \subset \mathbf{F}$ for any $i$, and thus $\mathbf{H}^1 \vee \cdots \vee \mathbf{H}^n \subset \mathbf{F}$. Since in this case the enlarged filtration $\mathbf{G} \stackrel{def}{=} \mathbf{H}^1 \vee \cdots \vee \mathbf{H}^n \vee \mathbf{F}$ coincides with $\mathbf{F}$, we shall assume in what follows that

$\tau_1, \ldots, \tau_n$ are $\mathbf{G}$-stopping times (this notational convention will prove useful in what follows).

Our goal is to examine the relationship between the $\mathbf{G}$-hazard processes of stopping times $\tau_i$ and the $\mathbf{G}$-hazard process of their minimum. To avoid discussion of cases of minor interest, we shall focus on the case when the stopping time $\tau_i$ are totally inaccessible, so that $\tau = \min(\tau_1, \ldots, \tau_n)$ is also a totally inaccessible stopping time with respect to $\mathbf{G}$. We borrow from Duffie [18] the following result (see Lemma 1 in [18]).

**Lemma 7.2** *Let $\tau_i$, $i = 1, \ldots, n$, be $\mathbf{G}$-stopping times such that $P(\tau_i = \tau_j) = 0$ for $i \neq j$. Then the $\mathbf{G}$-martingale hazard process $\Lambda$ of $\tau = \min_{i=1,\ldots,n} \tau_i$ is equal to the sum of $\mathbf{G}$-martingale hazard processes $\Lambda^i$ stopped at $\tau$, more precisely, $\Lambda_{t\wedge\tau} = \sum_{i=1}^n \Lambda^i_{t\wedge\tau}$ for every $t \in I\!R_+$.*

PROOF: By assumption, for any $i = 1, \ldots, n$, the process $M^i_t = N^i_t - \Lambda^i_{t\wedge\tau_i}$ is a $\mathbf{G}$-martingale. Therefore, by the well-known properties of martingales the stopped process[13]

$$(M^i_t)^\tau = N^i_{t\wedge\tau} - \Lambda^i_{t\wedge\tau_i\wedge\tau} = N^i_{t\wedge\tau} - \Lambda^i_{t\wedge\tau}$$

also follows a $\mathbf{G}$-martingale for any fixed $i$. On the other hand, since $P(\tau_i = \tau_j) = 0$ for $i \neq j$, we have

$$\sum_{i=1}^n N^i_{t\wedge\tau} = N_t = 1\!\!1_{\{\tau \leq t\}}.$$

Therefore, the process

$$N_t - \sum_{i=1}^n \Lambda^i_{t\wedge\tau} = \sum_{i=1}^n (M^i_t)^\tau$$

obviously follows a $\mathbf{G}$-martingale, as a sum of $\mathbf{G}$-martingales. We conclude that the $\mathbf{G}$-martingale hazard process $\Lambda$ of $\tau$ satisfies $\Lambda_{t\wedge\tau} = \sum_{i=1}^n \Lambda^i_{t\wedge\tau}$ for every $t \in I\!R_+$. □

[13] It is essential to assume that $\tau_i$, $i = 1 \ldots,$, are stopping time with respect to $\mathbf{G}$ so that $\tau$ is also a $\mathbf{G}$-stopping time. If $\tau$ is merely a random time the stopped process needs not to be a martingale.

The striking feature of Lemma 7.2 is that the **G**-martingale hazard process of $\tau$ can be calculated without the knowledge the joint law of stopping times $\tau_1, \ldots, \tau_n$. It should thus be observed that in order to make use of the notion of a **G**-martingale hazard process $\Lambda$ we need to show in addition that $\Lambda$ actually possesses required probabilistic properties. For instance, it would be useful to know whether

$$P(\tau > s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} E\big(e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{G}_t\big) \tag{89}$$

for $t \leq s$. Combining Lemma 7.2 with Corollary 5.4, we get immediately the following result which gives a partial answer to the last question.

**Proposition 7.1** *Let $\tau_i$, $i = 1, \ldots, n$, be **G**-stopping times such that $P(\tau_i = \tau_j) = 0$ for $i \neq j$. Assume that each stopping time $\tau_i$ admits an absolutely continuous **G**-martingale hazard process $\Lambda^i_t = \int_0^t \lambda^i_u \, du$. If the process $V$ given by the formula*

$$V_t = E\big(e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{G}_t\big) = E\big(e^{-\sum_{i=1}^n \int_t^s \lambda^i_u \, du} \,|\, \mathcal{G}_t\big), \quad \forall\, t \in [0, s], \tag{90}$$

*is continuous at $\tau = \min_{i=1,\ldots,n} \tau_i$, then for any $t < s$ we have*

$$P(\tau > s \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} E\big(e^{\Lambda_t - \Lambda_s} \,|\, \mathcal{G}_t\big) = 1\!\!1_{\{\tau > t\}} E\big(e^{-\sum_{i=1}^n \int_t^s \lambda^i_u \, du} \,|\, \mathcal{G}_t\big).$$

At the first glance Proposition 7.1 seems to be a very useful and powerful result, since apparently it covers both the case of independent stopping times and dependent stopping times. Notice, however, that we face here a rather delicate issue of checking the continuity of $V$ at $\tau$. As we shall show below, this condition is rarely satisfied when $\tau$ is the minimum of several stopping times. Furtheremore, we are dealing here with **G**-martingale hazard processes of **G**-stopping times; this notion is not useful when a given stopping time is **G**-predictable (see example in Section 7.2). It seems to us that the number of circumstances when Proposition 7.1 can be effectively applied is in fact very limited. One of them is examined in the following example, in which random times $\tau_i$ are conditionally independent, given the filtration **F**.

**Example 7.1** Let $\psi_i$, $i = 1, 2$, be two **F**-adapted increasing stochastic processes defined on some probability space $(\Omega, \mathcal{G}, P)$. We set

$$\begin{aligned} \tau_i &= \inf\{\, t \in I\!\!R_+ \,:\, \Psi^i_t \geq -\ln(\xi^i) \,\} \\ &= \inf\{\, t \in I\!\!R_+ \,:\, \int_0^t \psi^i_u \, du \geq -\ln(\xi^i) \,\}, \end{aligned} \tag{91}$$

where $\xi^1$, $\xi^2$ are mutually independent random variables, which are also independent of processes $\psi^i$, $i = 1, 2$, and are uniformly distributed on the unit interval $[0, 1]$. For each $i$, the enlarged filtration $\mathbf{G}^i = \mathbf{H}^i \vee \mathbf{F}$ satisfies $\mathcal{G}_t^i = \mathcal{F}_t \vee \mathcal{H}_t^i \subset \mathcal{F}_t \vee \sigma(\xi^i)$ for every $t$.

From Section 7.1 we know that the process $\Psi^i$ represents the $\mathbf{F}$-hazard process of $\tau_i$. In particular, for any $\mathcal{F}_s$-measurable random variable $Y$ we have for every $t \le s$ (cf. (87))

$$E(\mathbb{1}_{\{\tau_i > s\}} Y \mid \mathcal{G}_t^i) = \mathbb{1}_{\{\tau_i > t\}} E\big(Y e^{-\int_t^s \psi_u^i \, du} \mid \mathcal{F}_t\big). \tag{92}$$

In terms of the martingale characterization, the process $\Psi^i$ is the $(\mathbf{F}, \mathbf{G}^i)$-martingale hazard process of a random time $\tau_i$. Also, $\tau_i$ is a totally inaccessible stopping time with respect to $\mathbf{G}^i$, and the continuity condition of Corollary 5.4 is satisfied. Indeed, for any fixed $s > 0$, the process

$$V_t^i \stackrel{def}{=} E\big(e^{\Psi_t^i - \Psi_s^i} \mid \mathcal{G}_t^i\big) = E\big(e^{\Psi_t^i - \Psi_s^i} \mid \mathcal{F}_t\big), \quad \forall\, t \in [0, s],$$

is obviously continuous at $\tau_i$. We conclude that for any $t \le s$

$$\begin{aligned} P(\tau_i > s \mid \mathcal{G}_t^i) &= \mathbb{1}_{\{\tau_i > t\}} E\big(e^{-\int_t^s \psi_u^i \, du} \mid \mathcal{G}_t^i\big) \\ &= \mathbb{1}_{\{\tau_i > t\}} E\big(e^{-\int_t^s \psi_u^i \, du} \mid \mathcal{F}_t\big). \end{aligned} \tag{93}$$

To examine simultaneously $\tau_1$ and $\tau_2$ as stopping times, we introduce the filtration $\mathbf{G}$ by setting $\mathbf{G} = \mathbf{F} \vee \mathbf{H}^1 \vee \mathbf{H}^2$. Then $\tau_1$, $\tau_2$, as well as $\tau = \min(\tau_1, \tau_2)$ are $\mathbf{G}$-stopping times.

Let us observe that it is not quite obvious that the process $\Psi^i$ is the $\mathbf{G}$-martingale hazard process of a $\mathbf{G}$-stopping time $\tau_i$. Indeed, we know that $\Psi^i$ is a $\mathbf{G}$-adapted continuous process such that $N_t^i - \Psi_{t \wedge \tau_i}^i$ is a $\mathbf{G}^i$-martingale. It is thus enough to show that $N_t^i - \Psi_{t \wedge \tau_i}^i$ is also a $\mathbf{G}$-martingale. Let us consider, for instance, $i = 1$. The random variable $N_t^1 - \Psi_{t \wedge \tau_1}^1$ is $\mathcal{G}_t$-measurable. It is thus enough to check that for any $t \le s$

$$E(N_s^1 - \Psi_{s \wedge \tau_1}^1 \mid \mathcal{G}_t) = E(N_s^1 - \Psi_{s \wedge \tau_1}^1 \mid \mathcal{G}_t^1).$$

Notice that the $\sigma$-fields $\mathcal{G}_s^1$ and $\mathcal{H}_t^2$ are conditionally independent given $\mathcal{G}_t^1$. Consequently,

$$E(N_s^1 - \Psi_{s \wedge \tau_1}^1 \mid \mathcal{G}_t) = E(N_s^1 - \Psi_{s \wedge \tau_1}^1 \mid \mathcal{G}_t^1 \vee \mathcal{H}_t^2) = E(N_s^1 - \Psi_{s \wedge \tau_1}^1 \mid \mathcal{G}_t^1).$$

Since we have shown that $\Psi^1$ is the $\mathbf{G}$-martingale hazard process of $\tau^1$, we have (under mild assumption on $\mathcal{G}_s$-measurable random variable $Y$)

$$E(\mathbb{1}_{\{\tau_1>s\}}Y\,|\,\mathcal{G}_t)=\mathbb{1}_{\{\tau_1>t\}}\,E\big(Ye^{\Psi^1_t-\Psi^1_s}\,|\,\mathcal{G}_t\big).$$

In particular, we have for any $t\le s$ (cf. (93))

$$P(\tau_1>s\,|\,\mathcal{G}_t)=\mathbb{1}_{\{\tau_1>t\}}\,E\big(e^{\Psi^1_t-\Psi^1_s}\,|\,\mathcal{G}_t\big)=\mathbb{1}_{\{\tau_i>t\}}\,E\big(e^{-\int_t^s\psi^i_u\,du}\,|\,\mathcal{F}_t\big)$$

since the process

$$\widetilde{V}^1_t\stackrel{def}{=}E\big(e^{\Psi^1_t-\Psi^1_s}\,|\,\mathcal{G}_t\big)=E\big(e^{\Psi^1_t-\Psi^1_s}\,|\,\mathcal{F}_t\big),\quad\forall\,t\in[0,s],$$

is continuous at $\tau_1$.

In view of Lemma 7.2, the $\mathbf{G}$-martingale hazard process $\Psi$ of $\tau$, when stopped at $\tau$, is the sum of $\mathbf{G}$-martingale hazard processes $\Psi^i$, $i=1,2$, associated with $\mathbf{G}$-stopping times $\tau_i$, $i=1,2$, also stopped at $\tau$. We have for $t\le s$

$$\begin{aligned}P(\tau>s\,|\,\mathcal{G}_t)&=\mathbb{1}_{\{\tau>t\}}\,E\big(e^{\Psi_t-\Psi_s}\,|\,\mathcal{G}_t\big)\\&=\mathbb{1}_{\{\tau>t\}}\,E\big(e^{-\int_t^s(\psi^1_u+\psi^2_u)\,du}\,|\,\mathcal{F}_t\big).\end{aligned}\tag{94}$$

In should be stressed that the last formula is a consequence of the assumption that the underlying random variables $\xi^1$ and $\xi^2$ are independent. The case of dependent random variables $\xi^1$ and $\xi^2$ is much more involved; let us only observe that we cannot expect formula (94) to hold in this case. Indeed, it seems plausible that the $\mathbf{G}$-martingale hazard process of $\tau_1$ will have a jump at $\tau_2$ on the set $\{\tau_2<\tau_1\}$, and conversely, the $\mathbf{G}$-martingale hazard process of $\tau_2$ will be discontinuous at $\tau_1$ on the set $\{\tau_1<\tau_2\}$. Consequently, one may conjecture that the sum of these processes will have a discontinuity at $\tau$, and thus it will not be possible to use the $\mathbf{G}$-martingale hazard process of $\tau$ to directly represent the survival probability $P(\tau>s\,|\,\mathcal{G}_t)$ through a counterpart of formula (94).

At the intuitive level, if the underlying random variables $\xi^1$ and $\xi^2$ are not independent, the observed occurence of $\tau_2$ ($\tau_1$, resp.) has a sudden impact on our assessments of the likelihood of the occurence of $\tau_1$ ($\tau_2$, resp.) in a given time interval in the future. A very special case of such a situation, when $\xi^1=\xi^2$, was examined in Section 7.2. The general case remains, to our knowledge, an open problem.

**Remark 7.2** Alternatively, we may check that $\Psi^1$ is also the $(\mathbf{G}^2, \mathbf{G})$-martingale hazard process of $\tau_1$. Since $\Psi^1$ is a continuous $\mathbf{G}^2$-adapted process and $\mathbf{G} = \mathbf{G}^2 \vee \mathbf{H}^1$, it is enough to verify that $\Psi^1$ coincides with the $\mathbf{G}^2$-hazard process of $\tau_1$, or equivalently, that

$$P(\tau_1 > t \,|\, \mathcal{G}_t^2) = e^{-\Psi_t^1}, \quad \forall\, t \in I\!R_+.$$

The last equality is clear, however, since the $\sigma$-fields $\mathcal{G}_t^1$ and $\mathcal{H}_t^2$ are conditionally independent given $\mathcal{F}_t$, and thus (the event $\{\tau_1 > t\}$ belongs, of course, to $\mathcal{G}_t^1$)

$$P(\tau_1 > t \,|\, \mathcal{G}_t^2) = P(\tau_1 > t \,|\, \mathcal{F}_t \vee \mathcal{H}_t^2) = P(\tau_1 > t \,|\, \mathcal{F}_t) = e^{-\Psi_t^1}.$$

Of course, a similar property holds for the $\mathbf{G}$-stopping time $\tau_2$ and the filtration $\mathbf{G}^1$.

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