# THE PERIODOGRAM OF AN I.I.D SEQUENCE

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ABSTRACT. Periodogram ordinates of a Gaussian white noise computed at Fourier frequencies are well known to form an i.i.d. sequence. This is no longer true in the non Gaussian case. In this paper, we developp a full theory for weighted sums of non linear functionals of the periodogram of an i.i.d. sequence. We prove that these sums are asymptotically Gaussian under conditions very close to those which are sufficient in the Gaussian case, and that the asymptotic variance differs from the Gaussian case by a factor proportional to the fourth cumulant of the white noise. An important consequence is a fonctional central limit theorem for the spectral empirical measure.

## 1. Introduction

Let $(Z_t)_{t\in\mathbb{Z}}$ be a white noise with unit variance, *i.e.* an i.i.d sequence such that $\mathbb{E}[Z_0] = 0$ and $\mathbb{E}[Z_0^2] = 1$. Define the discrete Fourier transform and the periodogram as

$$rd_n^Z(x) = (2\pi n)^{-1/2} \sum_{t=1}^{n} Z_t e^{itx} \quad \text{and} \quad I_n^Z(x) = |d_n^Z(x)|^2.$$

The Fourier frequencies are usually defined as $x_k = 2\pi k/n$, $1 \le k \le \tilde{n}$ where $\tilde{n} = [(n-1)/2]$ (the dependency with respect to $n$ will be omitted). It is a well known fact that if the variables $Z_t$ are moreover Gaussian, then the periodogram ordinates computed at Fourier frequencies are independent and $2\pi I_n^Z(x_k)$ has a $\Gamma(1,1)$ distribution. The $\Gamma(a,\lambda)$ distribution is the distribution with density function $\Gamma(a)^{-1}\lambda^a x^{a-1}e^{-\lambda x}$ with respect to Lebesgues measure on $\mathbb{R}^+$ (where $\Gamma$ is the Gamma function). Gaussianity and the specific choice of the Fourier frequencies are the fundamental reasons for this independence. Let $0 \le k < j < \tilde{n}$.

$$\mathbb{E}[d_n^Z(x_k) d_n^Z(\pm x_j)] = \frac{1}{2\pi n} \sum_{t=1}^{n} e^{it(x_k \pm x_j)} = 0.$$

The last sum vanishes because of the specific choice of the Fourier frequencies. This implies uncorrelatedness of the variables $d_n^Z(x_k)$, hence independence in the Gaussian case. This latter property no longer holds in the non Gaussian case. For instance, let $\kappa_4$ denote the fourth cumulant of $Z_0$. An easy computation yields, for $0 \le k < j < \tilde{n}$,

$$\operatorname{cov}(I_n^Z(x_k), I_n^Z(x_j)) = \frac{\kappa_4}{4\pi^2 n}.$$

The fourth cumulant of a standard Gaussian variable is 0, but it is not necessarily so for an arbitrary distribution. Nevertheless, the central limit theorem implies that for any fixed $u$, $d_n^Z(x_{k_1}), \cdots, d_n^Z(x_{k_1})$ are asymptotically independent, in the sense that the asymptotic distribution of the $2u$-dimensional vector

$$(Re\{d_n^Z(x_{k_1})\}, Im\{d_n^Z(x_{k_1})\}, \cdots, Re\{d_n^Z(x_{k_u})\}, Im\{d_n^Z(x_{k_u})\})$$

is the $2u$-dimensional standard Gaussian distribution, *i.e.* the distribution of $2u$ i.i.d. $\mathcal{N}(0,1)$ random variables. This implies that $2\pi I_n(x_{k_1}), \cdots, 2\pi I_n(x_{k_u})$ are asymptotically independent exponentials. Anyhow, statistics of interest seldom involve a fixed finite number number of periodogram ordinates. Among important problems, we can mention the following.

**Asymptotic distribution of the maximum.** In the Gaussian case, $M_n = 2\pi \max_{1\leq k\leq \tilde{n}} I_n(x_k)$ has a $\Gamma(\tilde{n}, 1)$ distribution. Thus $\lim_{n\to\infty} \mathbb{P}(M_n - \log(\tilde{n}) \leq x) = e^{-e^{-x}}$ (the standard Gumbel distribution). Davis and Mikosch (1999) [6] have shown that this asymptotic property still holds true in the non Gaussian case.

**Weighted sums of functionals of the periodogram** Consider real numbers $\beta_{n,k}$ such that $\sum_{k=1}^{\tilde{n}} \beta_{n,k}^2 1$ and a function $\phi$ and define

$$S_n^Z(\phi) = \sum_{k=1}^{\tilde{n}} \beta_{n,k}\phi(I_n(x_k)).$$

In the Gaussian case, such a sum is asymptotically Gaussian under the necessary assumption that $\mathbb{E}[\phi^2(I_n(x_k))] < \infty$ and under the Lindeberg condition

$$\lim_{n\to\infty} \max_{1\leq k\leq \tilde{n}} |\beta_{n,k}| = 0.$$

The considerations above make one expect that this result still holds in the non Gaussian case. However, no general result of this kind is known. We recall now some previous results. In the case of a linear functional *i.e.* $\phi(x) = x$, if the weights $\beta_{n,k}$ are the value of a function $g$ at Fourier frequencies, it is known that $S_n^Z(\phi)$ is asymptotically Gaussian, with an asymptotic variance that depends on $\kappa_4$. For instance, if $\beta_{n,k} = \tilde{n}^{-1/2}$, $\tilde{n}^{-1/2}\sum_{k=1}^{\tilde{n}}(2\pi I_n(x_k) - 1)$ is asymptotically Gaussian with variance $1 + \kappa_4/2$. This can be proved by the method of cumulants, see Brillinger (1981) [4]. In the case of non linear functionals, very little is known. The first attempt to derive an asymptotice theory for such sums in the non-linear case is due to Chen and Hannan (1980) [5] in the case $\phi(x) = \log(x)$. They used the technique of Edgeworth expansions for triangular arrays of indepen dent random variables to compute the variance of $\tilde{n}^{-1/2}\sum_{k=1}^{\tilde{n}} \log(2\pi I_{n,k}) - \log(2) - \gamma$, where $\gamma$ is Euler's constant. If $q_n$ denotes the joint density of $Re\{d_n^Z(x_k)\}, Im\{d_n^Z(x_k)\}, Re\{d_n^Z(x_j)\}, Im\{d_n^Z(x_j)\})$, for $0 < k < j \leq \tilde{n}$, a second order Edgeworth expansion of $q_n$ yields (with the weights $\beta_{n,k}$ set equal to $\tilde{n}^{-1/2}$)

$$\mathrm{var}(S_n^Z(\log)) = \frac{\pi^2}{6} + \kappa_4/2 + O(n^{-1/2}).$$

Note that $\pi^2/6$ is exactly the variance in the Gaussian case. The main drawback of this method is that the computation of higher moments is extremely involved, (but this may be overcome), and that the existence of the joint density $q_n$ and the validity of its Edgeworth expansion require a regularity assumption on the distribution of $Z_0$, which nearly amounts to the existence of a density with respect to Lebesgue measure, and the necessity of which is not obvious. Let it be said, nevertheless, that in the case of non regular functionals, some regularity is needed since if the distribution of $Z_0$ has, say, a positive mass at zero, then the log-periodogram cannot be computed. Recently, Velasco (1999) [13] using the same method, proved a central limit theorem in the case of the function log and in the particular case where the number of non vanishing coefficients $\beta_{n,k}$ is negligible with respect to $n$. The asymptotic variance is then $\pi^2/6$, the same as in the Gaussian case. The central limit theorem is proved using the method of moments, and Velasco assumes that $\mathbb{E}[|Z_0|^s]$ is finite for all $s$. This is obviously a strong assumption that one would like to omit.

**Empirical spectral distribution function.** Another important and unsolved problem was to prove a functional central limit theorem for the empirical spectral measure, defined as

$$\hat{F}_n(x) = \tilde{n}^{-1} \sum_{k=1}^{\tilde{n}} \mathbf{1}_{[\mathbf{0},\mathbf{x}]}(\mathbf{2\pi I_n(x_k)}), \quad \mathbf{x} \geq \mathbf{0}.$$

Freedman and Lane (1980) [7] and Kokoszka and Mikosch (1998) [11] proved that under the only assumption that $\mathbb{E}(Z_t^2) < \infty$, $\sup_{x\geq 0} |\hat{F}_n(x) - F_1(x)|$ converges in probability to zero, where $F_1(x) = 1 - e^{-x}$ is the standard exponential cumulative distribution function. Kokoszka and Mikosch (1998) strengthened this result and proved convergence of the first three moments of $\tilde{n}^{-1/2}(\hat{F}_n(x) - F_1(x))$ under the natural assumptions of finiteness of the six first moments of $Z_0$ (but on the not so natural assumption that they all coincide with those of a $\mathcal{N}(0,1)$ distribution) and under the regularity assumption on the distribution of $Z_0$ mentioned above.

In this paper, using the ideas of Chen and Hannan (1980) and generalizing (and making more formal) the deep ideas of Velasco (1999), we present a full theory for weighted sums of (possibly) non linear functionals of the periodogram of an i.i.d. sequence, and we solve the abovementioned problems. We also bring a new tool to the study of this problem. While the cited authors used Edgeworth expansion of the joint density of a finite number of discrete Fourier transforms, which necessitates the regularity assumption, we use the results of Götze and Hipp (1978) [9] on Edgeworth expansions for moments of smooth functions. This allows, in the case of smooth functionals, to get rid of the regularity assumption on the distribution of $Z_0$. This, in its turn, allows to use truncation arguments to get also rid of the assumption of finite moments of all order to obtain a central limit theorem by means of the method of moments.

Before concluding this already long introduction, let us mention that in statistical applications, the quantity of interest is never $S_n^Z(\phi)$, but more likely $S_n^X(\phi)$, where $X$ is a process which admits a linear representation with the i.i.d. sequence $Z$, *i.e.* $X$ writes

$$X_t = \sum_{j\in\mathbb{Z}} a_j Z_{t-j},\ t \in \mathbb{Z},$$

where $(a_t)_{t\in\mathbb{Z}}$ is a sequence of real numbers such that $\sum_{t\in\mathbb{Z}} a_t^2 < \infty$. The spectral density of the process $X$ is then

$$f_X = (2\pi)^{-1} \left|\sum\nolimits_{j\in\mathbb{Z}} a_j e_j\right|^2,$$

where $e_j(x) = e^{ijx}$. If the coefficients $a_j$ are absolutely summable, then $f_X$ is continuous and the process $X$ is said weakly dependent. If the coefficients $a_j$ are not absolutely summable, then $f_X$ may not be continuous and even have singularities, in which case the process $X$ is usually said strongly dependent. The study of $S_n^X(\phi)$ is then based on the so-called Bartlett's decomposition (cf. Bartlett (1955) [1]), which consists in relating the periodogram of $X$ to that of $Z$ :

$$I_n^X(x) = 2\pi f_X(x) I_n^Z(x) + R_n(x),$$

where the superscript indicates the process with respect to which the periodogram is computed. Then one can write

$$S_n^X(\phi) = \sum_{k=1}^K \beta_{n,k}\phi(2\pi I_n^Z(x)) + T_n,$$
$$T_n = \sum_{k=1}^K \beta_{n,k}\{\phi(2\pi I_n^Z(x)) - \phi(2\pi f_X(y_k) I_n^Z(x))\}.$$

Under reasonable regularity assumptions, one can prove that $T_n$ tends to zero in probability, and the remaining task is to obtain a central limit theorem for $S_n^Z(\phi)$. The problem with this decomposition is that the remainder term $R_n$ is rather large, even if the coefficients $a_j$ decay very rapidly or are only finitely many. We will not give any statistical applications in this paper, but for the problem mentioned above, we can already say that our results yield a central limit theorem for the estimator of the estimation variance considered in Chen and Hannan (1980), and that we improve on Velasco (1999) since we prove that his central limit theorem holds if $Z_0$ has only a finite number of finite moments (the exact number depends on many parameters not specified here). Other applications for weak dependent linear processes are presented in Fay, Moulines and soulier (1999) [8] and an application to the estimation of the dependence coefficient of a fractional process is presented in Hurvich, Moulines and Soulier (1999) [10].

The rest of the paper is organized as follows. Since the technique of Edgeworth expansion is applied to the distribution of the discrete Fourier transforms, we first state a very general theorem for functionals of the Fourier transforms. Another motivation is that it can be applied to modifications of the periodogram such as tapered periodogram, not considered here for the sake of bervity, but that are very important in statistical applications, especially for long range dependent processes. In section 3, we apply this result to general linear functionals of the periodogram and in section 4, we state a functional central limit theorem for the empirical spectral distribution function. The proof of the main theorem, being very involved is split in several sections. The main technical tool, a moment expansion (Lemma 3) is stated in section 6 (Lemma 3) and proved in section 8. Even though it is just a technical lemma, we consider it as the actual main result of this paper, since all the other results easily derive from it, and because it offers the deepest insight into the dependence structure of periodogram ordinates at Fourier frequencies of a non Gaussian i.i.d. sequence.

## 2. Main result

Let $m$ be a fixed positive integer and define for all $n \geq 2m$, $K = [(n-m)/2m)]$. For $1 \leq k \leq K$, define the $2m$-dimensional vector

$$W_{n,k} = (2/n)^{1/2} \sum_{t=1}^{n} Z_t (\cos(tx_{m(k-1)+1}), \sin(tx_{m(k-1)+1}), \cdots, \cos(tx_{mk}), \sin(tx_{mk}))^T. \tag{1}$$

so that $2\pi \bar{I}_{n,k} = \|W_{n,k}\|^2/2$. In this section, we give conditions on a triangular array of functions $(\psi_{n,k})_{1 \leq k \leq n}$ to obtain a central limit theorem for sums $S_n := \sum_{k=1}^{n} \beta_{n,k} \psi_{n,k}(W_{n,k})$.

In the case of functions of non smooth functions, as mentioned in the introduction, a regularity assumption on the distribution of the white noise $Z_0$ is necessary.

**(A1)** There exists some real $r \geq 1$, such that $\int_{-\infty}^{+\infty} |\mathbb{E}(e^{itZ_0})|^r dt < \infty$.

Assumption **(A1)** ensures that $n^{-1/2}\sum_{t=1}^{n} Z_t$ has a density $q_n$ for all sufficiently large $n$ ($n \geq p$) and that this density converges uniformly to the standardized Gaussian distribution (see, for example, Bhattacharya and Rao (1976) [2], Theorem 19.1,p.189). It is a strengthening of the usual Cramer's condition which excludes "strongly lattice" variables. We now define the classes of admissible functions.

**Definition 1.** *$\mathcal{H}_\alpha$ is the set of measurable functions on $\mathbb{R}^{2m}$ such that $N_\alpha(\psi) < \infty$, where*

$$N_\alpha^2(\psi) = \int_{\mathbb{R}^{2m}} \psi^2(x)(1+|x|^2)^{-\alpha}dx.$$

It is easily seen that $\mathcal{H}_\alpha$, endowed with the norm $N_\alpha$ is a Hilbert space and that the set of compactly supported $C^\infty$ functions is dense in this space.

**Definition 2.** *For integers $\nu$ and $r$, let $\mathcal{S}_\nu^r$ be the space of $r$ times differentiable function on $\mathbb{R}^{2m}$ such that for all $2m$-tuples of integers $\beta = (\beta_1, \cdots, \beta_{2m})$ that satisfy $\beta_1 + \cdots + \beta_{2m} \leq r$,*

$$M_\nu(D^\beta\psi) < \infty,$$

*where $D^\beta$ denotes the partial derivative of $\psi$ of order $\beta_i$ with respect to the $i$-th component, and for any function $\phi$ on $\mathbb{R}^{2m}$,*

$$M_\nu(\phi) = \sup_{x\in\mathbb{R}^{2m}} \frac{|\phi(x)|}{1+|x|^\nu}.$$

The notation $M_\nu$ comes from Götze and Hipp (1978) [9]. For convenience, we introduce the following notation. For $\psi \in \mathcal{S}_\nu^r$, denote

$$M_{\nu,r}(\psi) = \sum_{\beta_1+\cdots+\beta_{2m}\leq r} M_\nu(D^\beta\psi). \tag{2}$$

Obviously, the space $\mathcal{S}_\nu^0$ is included in $\mathcal{H}_\alpha$ for any $\alpha > \nu + m$. For $\psi \in \mathcal{H}_\alpha$, we can also define

$$||\psi||^2 = \mathbb{E}[\psi^2(\xi)], \tag{3}$$

$$C_2(\psi, j) = \mathbb{E}[(\xi_j^2 - 1)\psi(\xi)], \tag{4}$$

where $\xi = (\xi_1, \cdots, \xi_{2m})^T$ denotes a $2m$-dimensional standard Gaussian vector. If $\psi \in \mathcal{H}_\alpha$ for some $\alpha > 0$, then $||\psi|| < \infty$. Moreover, there exists a constant $C_\alpha$ such that for all $\psi \in \mathcal{H}_\alpha$, $||\psi|| \leq C_\alpha N_\alpha(\psi)$. Also, for all $j = 1, \cdots, 2m$, $|C_2(\psi, j)| \leq \sqrt{2}||\psi||$. Recall now that the Hermite rank of a function $\psi$ such that $||\psi|| < \infty$ is the smallest integer $\tau$ such that there exists a polynomial $P$ of degree $\tau$ with $\mathbb{E}[P(\xi)\psi(\xi)] \neq 0$.

In this section, for the sake of simplicity, only functions of Hermite rank at least 2 will be considered. A very frequently verified sufficient condition for a function $\psi$ to have Hermite rank at least 2 is $\mathbb{E}[\psi(\xi)] = 0$ and $\psi$ is componentwise even.

For a triangular array of functions $(\psi_{n,k})_{1\le k\le K}$ such that $\|\psi_{n,k}\| < \infty$, and for a triangular array of reals $(\beta_{n,k})_{1\le k\le K}$, define

$$S_n = \sum_{k=1}^{K} \beta_{n,k}\psi_{n,k}(W_{n,k}).$$

The assumptions needed to prove the asymptotic normality of $S_n$ are now stated.

**(A2)** $(\beta_{n,k})_{1\le k\le K}$ is a triangular array of real numbers such that $\sum_{k=1}^{K} \beta_{n,k}^2 = 1$ and

$$\lim_{n\to\infty} \max_{1\le k\le K} |\beta_{n,k}| = 0.$$

**(A3)** There exists a real $\sigma > 0$ such that

$$\lim_{n\to\infty} \sum_{k=1}^{K} \beta_{n,k}^2 \|\psi_{n,k}\|^2 = \sigma^2.$$

**(A4)** There exists a real $\tau$ such that

$$\lim_{n\to\infty} n^{-1} \sum_{1\le k<l\le K} \beta_{n,k}\beta_{n,l} \sum_{i,j=1,\cdots,2m} C_2(\psi_{n,k}, i) C_2(\psi_{n,l}, j) = \tau$$

and $\sigma^2 + \kappa_4\tau/4 \ne 0$.

**(A5)**

$$\forall\epsilon > 0,\ \max_{1\le k\le K} |\beta_{n,k}| = O(\mu_n^{-1/2+\epsilon}).$$

where $\mu_n := \#\{k : 1 \le k \le K, \beta_{n,k} \ne 0\}$.

Assumption **(A2)** implies the Lindeberg-Levy smallness condition and together with **(A3)** is sufficient in the Gaussian case. Assumption **(A4)** is necessary in the non Gaussian case since it appears in the expansion of $\mathrm{var}(S_n)$. Assumption **(A5)** means that $\mu_n(\max_{1\le k\le K} |\beta_{n,k}|)^2$ is bounded by a slowly varying function of $\mu_n$. It holds when $\beta_{n,k}$ is defined as $g(y_k)/\left(\sum_{k=1}^{K} g^2(y_k)\right)^{1/2}$ for most "reasonable" functions $g$ (such as continuous functions on $[-\pi, \pi]$ or $g(x) = \log(x)$) and evenly spaced frequencies $y_k$, $1 \le k \le K$. This assumption does not seem necessary, but we cannot prove our result without it. See the comment after Theorem 1. The next assumption is necessary to replace eventually non smooth functions $\psi_{n,k}$ by smooth ones.

**(A6)** For all real $\epsilon > 0$, there exists a sequence of compactly supported $C^\infty$ functions $\psi^\epsilon_{n,k}$ with same support $\mathcal{K}_\epsilon$ and with Hermite rank 2 such that

$$\max_n \max_{1\le k\le K} \|\psi_{n,k} - \psi^\epsilon_{n,k}\| \le \epsilon,$$

$$\forall \epsilon > 0,\ \forall r \in \mathbb{N},\ \exists C_{r,\epsilon},\ M_{0,r}(\psi^\epsilon_{n,k}) \le C_{r,\epsilon},$$

and there exist a real $\sigma^2(\epsilon) > 0$ and a real $\tau(\epsilon)$ such that

$$\lim_{n\to\infty} \sum_{k=1}^{K} \beta^2_{n,k} \|\psi^\epsilon_{n,k}\|^2 = \sigma^2(\epsilon),$$

$$\lim_{n\to\infty} n^{-1} \sum_{1\le k<l\le} \beta_{n,k}\beta_{n,l} \sum_{i,j=1,\cdots,2m} C_2(\psi^\epsilon_{n,k}, i) C_2(\psi^\epsilon_{n,l}, j) = \tau(\epsilon).$$

Note that if **(A3)**, **(A4)** and **(A6)** hold, then $\lim_{\epsilon\to 0} \sigma^2(\epsilon) = \sigma^2$ and $\lim_{\epsilon\to 0} \tau(\epsilon) = \tau$.

**Theorem 1.** *Let $(Z_t)_{t\in\mathbb{Z}}$ be a unit variance white noise with finite moment of order $\mu$. Let $(\beta_{n,k})_{1\le k\le K}$ be a triangular array of reals and $(\psi_{n,k})_{1\le k\le K}$ be a triangular array of functions such that Assumptions* **(A2)**, **(A3)**, **(A4)**, **(A5)**, *and* **(A6)** *hold. Assume either*

- *For all $1 \le k \le K$, $\psi_{n,k} \in \mathcal{H}_\alpha$, assumption* **(A1)** *holds and $\mu \ge 2\alpha + 2$.*
- *For all $1 \le k \le K$, $\psi_{n,k} \in \mathcal{S}^2_\nu$ and $\mu \ge 4\nu \vee 4$.*

*Then the distribution of $S_n$ is asymptotically centered Gaussian with variance $\sigma^2 + \tau\kappa_4/4$. Moreover assumptions* **(A4)** *and* **(A5)** *are not necessary in the following cases.*

- *If $\kappa_4 = 0$ then Assumption* **(A4)** *is not necessary.*
- *If $\mu_n = o(n^{2/3})$ then Assumption* **(A5)** *is not necessary and Assumption* **(A4)** *holds with $\gamma = 0$.*
- *If for all $k \le K$, $C_2(\psi_{n,k}) = 0$, then Assumptions* **(A5)** *and* **(A4)** *are not necessary and thus the central limit theorem holds under the same assumption on the weights $\beta_{n,k}$ and with the same limit as in the Gaussian case.*

**Comments.** This result gives a better understanding of the differences between the Gaussian and the non Gaussian case. Recall that in the Gaussian case only assumption **(A2)** is necessary. Here, we need a stronger assumption on the functions considered, and also a restriction on the admissible weights. Note that assuption **(A1)** holds in the Gaussian case, so it cannot be considered as a restriction. The strengthened assumptions on the functions considered are not somehow necessary, since some conditions are needed to insure integrability of $\psi_{n,k}(d_{n,k})$. The conditions we impose are nearly minimal, and in the case of smooth functions, they are optimal in terms of the requirement on the moments of $Z_0$. Assumption **(A5)** is probably not necessary. As mentioned in the Theorem, it is indeed not necesary in some cases.

## 3. Non linear functionals of the periodogram

Since Theorem 1 is stated for arbitrary $m$, we can derive a central limit theorem for non linear functionals of the aggregated (or averaged, or pooled) periodogram. Let $m$ be a fixed integer and set $K = [(n-m)/2m]$. Define

$$\bar{I}_{n,k} = \sum_{s=(k-1)m+1}^{km} I_n(x_s), \quad 1 \leq k \leq K.$$

Let $\phi$ be a measurable function on $\mathbb{R}$ such that $\mathbb{E}[\phi^2(Y)] < \infty$ where $Y$ is a $\Gamma(m,1)$ random variable, or, equivalently, $Y$ is distributed as $|\xi|^2/2$, where $\xi$ denote a $2m$-dimensional standard Gaussian vector, and $|.|$ denotes the Euclidean norm. The following quantities are then well defined.

$$\gamma_m(\phi) = \mathbb{E}[\phi(|\xi|^2/2)], \tag{5}$$

$$\sigma_m^2(\phi) = \mathrm{var}(\phi(|\xi|^2/2)) = \mathbb{E}[\phi^2(|\xi|^2/2)] - \gamma_m^2(\phi), \tag{6}$$

$$C_m(\phi) = \mathbb{E}[(\xi_1^2 - 1)\phi(|\xi|^2/2)]. \tag{7}$$

Let $(\beta_{n,k})_{1\leq k \leq K}$ be a triangular array of real numbers such that **(A2)** holds. In the context of this section, assumption **(A3)** will hold automatically, while **(A4)** will be a consequence of the following assumption.

**(A7)** There exists a real $\gamma$ such that $\lim_{n\to} n^{-1} \sum_{k\neq l} \beta_{n,k}\beta_{n,l} = \gamma$.

Define finally

$$S_n(\phi) = \sum_{k=1}^{K} \beta_{n,k}\{\phi(2\pi \bar{I}_{n,k}) - \gamma_m(\phi)\} \tag{8}$$

**Theorem 2.** *Let $(Z_t)_{t\in\mathbb{Z}}$ be a unit variance white noise with finite moment of order $\mu$. Assume either*

Smooth case *$\phi$ is twice differentiable, there exists an integer $\nu$ such that*

$$\max_{x\in\mathbb{R}} \frac{|\phi(x)| + |\phi'(x)| + |\phi''(x)|}{1+|x|^{\nu}} < \infty$$

*and $\mu \geq 2\nu \vee 4$.*

Nonsmooth case *Assumption* **(A1)** *holds, there exists a positive integer $\alpha$ such that*

$$\int_{\mathbb{R}^{2m}} \phi^2(|x|^2)(1+|x|^2)^{-\alpha} dx < \infty$$

*and $\mu \geq 2\alpha + 2$.*

*Let $(\beta_{n,k})_{1\leq k\leq K}$ be a triangular array satisfying Assumptions* **(A2)**, **(A5)** *and* **(A7)**, *and such that $\sigma_m^2(\phi) + \gamma m^2 \kappa_4 C_m^2(\phi) \neq 0$. Then $S_n(\phi)$ converges in distribution to the standard Gaussian distribution with variance $\sigma_m^2(\phi) + \gamma m^2 \kappa_4 C_m^2(\phi)$. Assumptions* **(A7)** *and* **(A5)** *are not necessary in the following cases.*

- *If $\kappa_4 = 0$ then Assumption* **(A7)** *is not necessary.*
- *If $\mu_n = o(n^{2/3})$ then Assumption* **(A5)** *is not necessary and Assumption* **(A7)** *holds with $\gamma = 0$.*
- *If $C_m(\phi) = 0$, then Assumptions* **(A5)** *and* **(A7)** *are not necessary and thus the central limit theorem holds under the same assumption on the weights $\beta_{n,k}$ as in the Gaussian case and with the same limit $\sigma_m^2(\phi)$.*

**Proof of Theorem 2.** If $\phi$ satisfy the assumptions of Theorem 2, define, for $x \in \mathbb{R}^{2m}$, $\psi(x) = \phi(|x|^2/2) - \gamma_m(\phi)$ and $\psi_{n,k} = \psi$ for all $1 \leq k \leq K$. As mentionned above, $\psi$ has Hermite rank 2 since $\mathbb{E}[\psi(\xi)] = 0$ and $\psi$ is componentwise even. If the array $\beta_{n,k}$ satisfy Assumptions **(A2)** and **(A7)** then Assumptions **(A3)** and **(A4)** hold with $\sigma^2 = \sigma_m^2(\phi)$ and $\tau = 4m^2\gamma C_m^2(\phi)$. Under the assumptions of Theorem 2, $\phi$ can be approximated by a sequence of compactly supported $C^\infty$ function $\phi^\epsilon$, *i.e.*

$$\forall \epsilon > 0, \quad \mathbb{E}[(\phi(|\xi|^2/2) - \phi^\epsilon(|\xi|^2/2))^2] \leq \epsilon.$$

Define then $\psi^\epsilon(x) = \phi^\epsilon(|x|^2/2)$ and $\psi_{n,k} = \psi$ for all $n$ and $k$. It can be assumed, without loss of generality, that $\mathbb{E}[\phi(\xi)] = \mathbb{E}[\phi^\epsilon(|\xi|^2/2)] = 0$. Thus the functions $\psi_{n,k}$ and $\psi^\epsilon_{n,k}$ all have Hermite rank at least 2, and assumption **(A6)** holds. Thus, Theorem 2 follows from Theorem 1. Since the proof of Theorem 1 is based on the so-called method of moments, it is an immediate by-product that, under a relevant moment assumption, convergence of moments holds.

**Proposition 1.** *Let $q$ be an integer. Under the assumptions of Theorem 2, if moreover $\mathbb{E}[Z_0^{2q\vee 4}] < \infty$ in the smooth case, or $\mathbb{E}[Z_0^{q\alpha+2}] < \infty$ in the nonsmooth case, then*

$$\lim_{n\to\infty} \mathbb{E}[S_n^{2q}(\phi)] = \frac{2q!}{2^q q!}(\sigma_m^2(\phi) + \gamma m^2 \kappa_4 C_2^2(\phi))^q.$$

## 4. Functional central limit theorem for the empirical spectral measure

The empirical spectral distribution function is defined as

$$\hat{F}_n(x) = K^{-1} \sum_{k=1}^{K} \mathbf{1}_{[0,x]}(2\pi \bar{I}_{n,k}), \quad x \geq 0.$$

In the case $m = 1$, it has been shown by Freedman and Lane (1980) [7] and in Kokoszka and Mikosch (1998) [11] that under the only assumption that $\mathbb{E}(Z_t^2) < \infty$, $\sup_{x\geq 0} |\hat{F}_n(x) - F_1(x)|$ converges in probability to zero, where $F_1(x) = 1 - e^{-x}$ is the standard exponential cumulative distribution function. Kokoszka and Mikosch (1998) [11] also proved that if the distribution of $Z_0$ satisfies the Cramer condition **(A1)**, if $\mathbb{E}(|Z_t|^6) < \infty$ and the first 6 moments of $Z_0$ coincide with those of a standard normal variable, then $\lim_{n\to\infty} n^{s/2}\mathbb{E}[(\hat{F}_n(x) - F_1(x))^s] = 0$ for $s = 1, 3$ and $\lim_{n\to\infty} n\mathbb{E}[(\hat{F}_n(x) - F_1(x))^2] = 2F_1(x)(1 - F_1(x))$. But these author were unable to derive convergence in distribution of $\sqrt{n}(\hat{F}_n(x) - F_1(x))$ and ask if a functional central limit theorem can be proved. Applying Theorem 2, we prove here that under **(A1)** and a suitable moment condition, the functional central limit theorem holds, and that $n^q\mathbb{E}[(\hat{F}_n(x) - F_1(x))^{2q}]$ converges to $\{2F_1(x)(1 - F_1(x))\}^q$ under the only additional assumption that $\kappa_4 = 0$. Define $F_m(x) = ((m-1)!)^{-1} \int_0^x t^{m-1} e^{-x} dx$, the distribution function of the $\Gamma(m, 1)$ distribution. Define also $C_m(x) = (m+1)F_{m+1}(x) - F_m(x)$.

**Theorem 3.** *If Assumption* **(A1)** *holds and if $\mathbb{E}(|Z_0|^8) < \infty$, then $\sqrt{n}(\hat{F}_n(x) - F_m(x))$ converges in the space $\mathcal{D}([0, \infty[)$ of left-limited right-continuous (cadlag) functions on $[0, \infty)$ to the Gaussian process $G(x)$*

*with covariance function*

$$\mathbb{E}[G(x)G(y)] = 2mF_m(x \wedge y)(1 - F_m(x \vee y)) + m^2\kappa_4 C_m(x)C_m(y).$$

*If moreover* $\mathbb{E}(|Z_0|^q) < \infty$*, then*

$$\lim_{n\to\infty} n^{q/2}\mathbb{E}[(\hat{F}_n(x) - F_m(x))^q] = \begin{cases} 0 & \text{if } q \text{ is odd,} \\ \{2mF_m(x \wedge y)(1 - F_m(x \vee y)) + m^2\kappa_4 C_m(x)C_m(y)\}^{q/2} & \text{if } q \text{ is even.} \end{cases}$$

**Remarks.**

- If $\kappa_4 = 0$ then the limit process is the same as if $Z_t$ were Gaussian white noise, or, equivalently, if the periodogram ordinates $\bar{I}_{n,k}$ were i.i.d. random variables with $\Gamma(m,1)$ distribution (i.i.d. exponentials in the case $m = 1$). If enough moments of $Z_0$ are finite, the limiting moments are also the same as in the Gaussian case. Thus, the difference with the behaviour of an i.i.d. sequence appears only through the fourth cumulant.
- The proof of Theorem 3 is as usual split into two parts. The convergence of finite distribution is an obvious consequence of Theorem 2 and holds under finiteness of the fourth moment of $Z_0$ only. Tightness is proved using the criterion for empirical processes of Shao and Yu (1996) [12] and needs finiteness of the eighth moment of $Z_0$.

## 5. Proof of Theorem 1

Theorem 1 is proved by means of the method of moments and Edgeworth expansions. Thus the first step in its proof is to prove a central limit theorem in the case of smooth functions and when all the moments of $Z_0$ are finite.

**Proposition 2.** *Assume that* $(Z_t)_{t\in\mathbb{Z}}$ *is a unit variance white noise such that for all integers* $s$, $\mathbb{E}(|Z_0|^s) < \infty$*. Assume that for all* $1 \le k \le K$ $\psi_{n,k}$ *is compactly supported* $C^\infty$ *and*

$$\forall r \in \mathbb{N}, \exists C_r, \forall n, \forall k \le K,\ M_{0,r}(\psi_{n,k}) \le C_r. \tag{9}$$

*Let* $(\beta_{n,k})_{1\le k\le K}$ *be a triangular array of reals such that Assumptions* **(A2)**, **(A5)**, **(A3)** *and* **(A4)** *hold. Then* $S_n$ *is asymptotically Gaussian with variance* $\sigma^2 + \tau\kappa_4/4$.

We must now relax the assumption that $Z_0$ has finite moments of all orders. Define $Z_t^{(M)} = \sigma_M^{-1} Z_t \mathbf{1}_{\{|\mathbf{Z_t}|\leq \mathbf{M}\}}$ and with $\sigma_M^2 = \mathbb{E}((Z_t \mathbf{1}_{\{|\mathbf{Z_t}|\leq \mathbf{M}\}})^{\mathbf{2}})$. Without loss of generality, we can assume that for all $M$, $\mathbb{E}(Z_t^{(M)}) = 0$, since we will compute discrete Fourier transforms at Fourier frequencies. Define $W_{n,k}^{(M)}$ in the same way as $W_{n,k}$, replacing $Z$ by $Z^{(M)}$.

**Lemma 1.** *Let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of zero-mean random variables with finite moment of order 4. Let $(\beta_{n,k})_{1\leq k\leq K}$ be a triangular array of real numbers such that $\sum_{k=1}^K \beta_{n,k}^2 = 1$. Assume that $(\psi_{n,k})_{1\leq k\leq K}$ is a triangular array of compactly supported $C^2$ functions with same support $\mathcal{K}$ and that there exists a constant $C$ such that*

$$\forall n, \forall k \leq K,\ M_{0,2}(\psi_{n,k}) \leq C. \tag{10}$$

*Then*

$$\lim_{M\to\infty} \limsup_{n\to\infty} \mathbb{E}\left(\sum_{k=1}^K \beta_{n,k}\{\psi_{n,k}(W_{n,k}) - \psi_{n,k}(W_{n,k}^{(M)})\}\right)^2 = 0.$$

**Proposition 3.** *Assume that $(Z_t)_{t\in\mathbb{Z}}$ is a unit variance white noise such $\mathbb{E}(|Z_0|^4) < \infty$. Assume that for all $1 \leq k \leq K$ $\psi_{n,k}$ is compactly supported $C^\infty$ and (9) holds. Let $(\beta_{n,k})_{1\leq k\leq K}$ be a triangular array of reals such that Assumptions* **(A2)**, **(A5)**, **(A3)** *and* **(A4)** *hold. Then $S_n$ is asymptotically Gaussian with variance $\sigma^2 + \tau\kappa_4/4$.*

**Proof of Proposition 3.** Define $S_n^{(M)} = \sum_{k=1}^K \beta_{n,k}\psi_{n,k}(W_{n,k}^{(M)})$. Applying Proposition 2 and Lemma 1, we get

$$\forall M \in \mathbb{N},\ S_n^{(M)} \xrightarrow{(d)} \mathcal{N}(0, \sigma^2(M)), \tag{11}$$

$$\lim_{M\to\infty} \limsup_{n} \mathbb{E}(S_n(M) - S_n)^2 = 0. \tag{12}$$

where $\sigma^2(M) = \lim_{n\to\infty} s_n^2(M)$ and $s_n^2(M)$ is defined as $s_n^2$ with the fourth cumulant $\kappa_4^{(M)}$ of $Z_0^{(M)}$ instead of $\kappa_4$. $\sigma^2(M)$ is well defined because of assumptions **(A3)** and **(A4)**. Moreover,

$$|s_n^2(M) - s_n^2| \leq C|\kappa_4^{(M)} - \kappa_4| \sum_{k=1}^K \beta_{n,k}^2 \|\psi_{n,k}\|^2.$$

This last sum is bounded because $\max_{1\leq k\leq K} \|\psi_{n,k}\| < \infty$ under assumption (9). Thus $\lim_{M\to\infty} \sigma^2(M) = \sigma^2$. Theorem 4.2 in Billingsley (1968) [3] concludes the proof of Proposition 3.

To conclude the proof of Theorem 1, there only remains to replace the sequence $\psi_{n,k}$ by a sequence of smooth functions.

**Lemma 2.** *Assume either*

- **(A1)** *holds, there exists a positive integer $\alpha$ and a constant $C$ such that for all $n$ and $1 \le k \le K$, $\psi_{n,k} \in \mathcal{H}_\alpha$ and $N_\alpha(\psi_{n,k}) \le C$, and $\mathbb{E}(|Z_t|^{2\alpha+2}) < +\infty$ ;*
- *there exists an integer $\nu$ and a constant $C$ such that for all $n$ and $1 \le k \le K$, $\psi_{n,k} \in \mathcal{S}_\nu^2$ and $M_{\nu,2}(\psi_{n,k}) \le C$, and $\mathbb{E}(|Z_t|^{2\nu} \vee 4) < +\infty$.*

*Then, for all triangular array of integers $\beta_{n,k}$ such that $\sum_{k=1}^K \beta_{n,k}^2 = 1$, for large enough $n$,*

$$\limsup_n \mathbb{E}\Big[\Big(\sum_{k=1}^K \beta_{n,k}\psi_{n,k}(W_{n,k})\Big)^2\Big] \le \sum_{k=1}^K \beta_{n,k}^2 \|\psi_{n,k}\|^2.$$

We can now conclude the proof of Theorem 1. Using the notations of assumption **(A6)**, denote $S_n(\epsilon) = \sum_{k=1}^K \beta_{n,k}\psi_{n,k}^\epsilon(W_{n,k})$. Applying Proposition 3 and Lemma 2, we have

$$\forall \epsilon > 0,\ S_n(\epsilon) \xrightarrow{(d)} \mathcal{N}(0, \sigma^2(\epsilon)),$$
$$\lim_{\epsilon\to 0} \limsup_n \mathbb{E}(S_n(\epsilon) - S_n)^2 = 0,$$
$$\lim_{\epsilon\to 0} \sigma^2(\epsilon) = \sigma^2.$$

We conclude as above by applying Theorem 4.2 in Billingsley (1968) [3].

## 6. Proof of Proposition 2 and of Lemmas 1 and 2

The proofs of Proposition 2 and of Lemmas 1 and 2 are based on a moment expansion for functions of the periodogram.

**Lemma 3.** *Let $s \le d$ be two integers. Let $k = (k_1, \cdots, k_d)$ be a $d$-tuple of pairwise distinct integers. Let $\phi_1, \cdots, \phi_d$ be $d$ functions defined on $\mathbb{R}^{2m}$. Assume that one of the following assumption holds.*

**(BR)** **(A1)** *holds and for all $i = 1, \cdots, d$, $N_{\alpha_i}(\phi_i) < \infty$ for some integers $\alpha_1, \cdots, \alpha_d$ and $\mathbb{E}[|Z_0|^\alpha] < \infty$ with $\alpha = \alpha_1 + \cdots + \alpha_d + 2$.*

**(GH)** *Denote* $r = (s-2md)^+ + 2$. *Let* $\nu_1, \cdots, \nu_d$ *be non negative integers and denote* $\nu = (\nu_1 + \cdots + \nu_d) \vee (s+2)$. *For all* $i = 1, \cdots, d$, $\phi_i \in \mathcal{S}^r_{\nu_i}$ *and* $\mathbb{E}(|Z_0|^\nu) < \infty$.

*Let* $\tau_i$ *be the Hermite rank of* $\phi_i$, $1 \le i \le s$ *and* $\tau = \inf_{1 \le i \le s} \tau_i$.

- *If* $\tau = 2$ *or* $3$ *then*

$$\mathbb{E}[\prod_{i=1}^d \phi_i(W_{n,k_i})] = n^{-s/2} \frac{s! \kappa_4^{s/2}}{2^{3s/2}(s/2)!} \sum_{j_1, \cdots, j_s = 1 \cdots 2m} \prod_{j=1}^s C_2(\phi_i, j_i) \prod_{i=s+1}^d \mathbb{E}[\phi_i(\xi)] \mathbf{1}_{\{\mathbf{s} \in \mathbf{2}\mathbb{N}\}}$$
$$+ \sum_{r=[(2s+2)/3]}^s n^{-r/2} \mathbb{F}_{r,k}(\phi_1, \cdots, \phi_d) + n^{-s/2} r_n(\phi_1, \cdots, \phi_d, k),$$
$$|\mathbb{F}_{r,k}(\phi_1, \cdots, \phi_d)| \le C \prod_{i=1}^d \|\phi_i\| \Delta_r(k),$$

  *where* $\Delta_r$ *is uniformly bounded by one and vanishes outside a finite union of subspaces of* $\mathbb{R}^d$, *the greatest dimension of which is strictly less than* $d + (r-s)/2$.
- *If* $\tau \ge 4$, *then*

$$\mathbb{E}[\prod_{i=1}^d \phi_i(W_{n,k_i})] = \prod_{i=1}^d \mathbb{E}[\phi_i(\xi)] + n^{-s/2} r_n(\phi_1, \cdots, \phi_d, k). \tag{13}$$

$\epsilon_n$ *is a sequence which depends only on* $d$, $s$, $\alpha_1, \cdots, \alpha_d$ *or* $\nu_1, \cdots, \nu_d$ *and the distribution of* $Z_0$ *and such that* $\lim_n \epsilon_n = 0$. *The following bounds hold for* $r_n$ *:*

- *if assumption* **(BR)** *holds :*

$$|r_n(\phi_1, \cdots, \phi_d, k)| \le \prod_{i=1}^d N_{\alpha_i}(\phi_i), \tag{14}$$

- *if assumption* **(GH)** *holds :*

$$|r_n(\phi_1, \cdots, \phi_d, k)| = \prod_{i=1}^d M_{\nu_i, r}(\phi_i). \tag{15}$$

**Remarks.**

- The constants involved in the above bounds are uniform wrt $n$ and $k_1, \cdots, k_d$ but depend on $d$.

- In the context of Theorem 2 or 3, Lemma 3 is used with $\phi_1 = \cdots = \phi_s = \psi$ for some function $\psi$ such that $\|\psi\| < \infty$ and $C_2(\psi, 1) = \cdots = C_2(\psi, 2m) := C_2(\psi)$. Then the first term in the expansion of $\mathbb{E}[\prod_{i=1}^d \phi_i(W_{n,k_i})]$ becomes, if $s$ is even and $\tau \geq 2$,

$$n^{-s/2} \frac{s!(m^2 C_2^2(\psi)\kappa_4/2)^{s/2}}{(s/2)!} \prod_{j=s+1}^{d} \mathbb{E}[\phi_j(\xi)].$$

- In (13), the product vanishes if $s > 0$.
- The case $\tau = 0$ is included in the case $s = 0$.
- In view of Lemma 2, it is important that the bound (14) is explicit in terms of the norms $N_{\alpha_i}(\phi_i)$.

6.1. **Proof of Proposition 2.** The proof is based on the method of moments. Denote $Y_{n,k} = \psi(\bar{I}_{n,k})$ and $\sigma_{n,k}^2 = \mathbb{E}[\psi_{n,k}^2(\xi)]$. Recall that $\sum_{k=1}^K \beta_{n,k}^2 = 1$. Let $q \in \mathbb{N}$, $q \geq 2$.

$$\mathbb{E}(S_n^q) = \sum_{v=1}^{q} \sum\nolimits_{v,q}' \frac{q!}{q_1! \cdots q_v!} \frac{1}{v!} A_n(q_1, \cdots, q_v),$$

$$A_n(q_1, \cdots, q_v) = \sum\nolimits_{v,n}'' \prod_{i=1}^{v} \beta_{n,k_i}^{q_i} \mathbb{E}\left(\prod_{i=1}^{v} Y_{n,k_i}^{q_i}\right),$$

$\sum_{v,q}'$ extends on all $v$-tuples of positive integers $(q_1, \cdots, q_v)$ such that $q_1 + \cdots + q_v = q$ and $\sum_{v,n}''$ extends on all $v$-uplets $(k_1, \cdots, k_v)$ of pairwise distinct integers in the range $\{1, \cdots, K\}$.

For any $v$-tuple $(q_1, \cdots, q_v)$ such that $q_1 + \cdots q_v = q$, let $s$ be the number of indices $i$ such that $q_i = 1$ and $u$ be the number of indices $i$ such that $q_i = 2$. Denote $w = v - s - u$. If $w > 0$, we easily get that $A_n(q_1, \cdots, q_v) = o(1)$. Indeed, Assumption **(A6)** and Lemma 3 yield, with $b_n = \max_{1 \leq k \leq K} |\beta_{n,k}|$,

$$|A_n(q_1, \cdots, q_v)| \leq C b_n^q \mu_n^{v - s/2}.$$

Now, $w > 0$ implies that $v - s/2 = s/2 + u + w \leq q/2 - 1/2$. Thus, under assumption **(A5)**,

$$|A_n(q_1, \cdots, q_v)| \leq C(b_n^2 \mu_n)^{q/2} \mu_n^{-1/2} = o(1).$$

Consider now $(q_1, \cdots, q_v)$ a $v$-tuple such that $w = 0$, *i.e.* $s + u = v$ and $s + 2u = q$.

- If $s$ is even, Lemma 3 yields

$$\begin{aligned} A_n(q_1,\cdots,q_v) &= A_n(1,\cdots,1,2,\cdots,2) \\ &= n^{-s/2}\frac{s!\kappa_4^{s/2}}{2^{3s/2}(s/2)!}\sum\nolimits''_{v,n}\prod_{i=1}^{s}\left\{\beta_{n,k_i}\sum_{j_i=1}^{2m}C_2(\psi_{n,k_i},j_i)\right\}\prod_{i=s+1}^{v}\beta_{n,k_i}^2\sigma_{n,k_i}^2+\epsilon_n \\ |\epsilon_n| &\le C(b_n^2\mu_n)^{q/2}\mu_n^{-1}. \end{aligned}$$

- If $s$ is odd (when $w=0$ and $q$ is odd), Lemma 3 yields

$$|A_n(q_1,\cdots,q_v)| \le C(b_n^2\mu_n)^{q/2}\mu_n^{-1}.$$

The leading term in the expansion of $\mathbb{E}(S_n^q)$ for any even $q$ is thus, (note that $v=(q+s)/2$ and denote $t=s/2$),

$$\tilde{s}_{n,q} = \frac{q!}{(q/2)!2^{q/2}}\sum_{t=0}^{q/2}\binom{q/2}{t}\left(\frac{\kappa_4}{4n}\right)^t n^{-t}\sum\nolimits''_{t+q/2,n}\prod_{i=1}^{2t}\beta_{n,k_i}\sum_{j_i=1}^{2m}C_2(\psi_{n,k_i},j_i)\prod_{i=2t+1}^{v}\beta_{n,k_i}^2\sigma_{n,k_i}^2.$$

Denote

$$s_n^2 = \sum_{k=1}^{K}\beta_{n,k}^2\|\psi_{n,k}\|^2+\frac{\kappa_4}{4n}\sum_{1\le k\ne l\le K}\beta_{n,k}\beta_{n,l}\sum_{i,j=1,\cdots,2m}C_2(\psi_{n,k},i)C_2(\psi_{n,l},j).$$

$s_n^2$ is the leading term of $\mathbb{E}[S_n^2]$ and assumptions **(A3)** and **(A4)** imply that $\lim_{n\to\infty}s_n^2=\sigma^2+\tau\kappa_4/4$. Since $b_n=o(1)$ (Assumption **(A2)**), it also holds that

$$\begin{aligned} s_n^q &= \sum_{t=0}^{q/2}\binom{\frac{q}{2}}{t}\left(\sum_{k=1}^{K}\beta_{n,k}^2\sigma_{n,k}^2\right)^{\frac{q}{2}-t}\left(\frac{\kappa_4}{4n}\sum_{1\le k_1\ne k_2\le\mu_n}\beta_{n,k_1}\beta_{n,k_2}\sum_{j_1,j_2=1\cdots 2m}C_2(\psi_{n,k_1},j_1)C_2(\psi_{n,k_2},j_2)\right)^t \\ &= \left(\frac{q!}{2^{q/2}(q/2)!}\right)^{-1}\tilde{s}_{n,q}(1+O(b_n)), \end{aligned}$$

and finally

$$\lim_{n\to\infty}\mathbb{E}(S_n^q) = \frac{q!}{2^{q/2}(q/2)!)}(\sigma+\tau\kappa_4/4)^q,$$

which concludes the proof of Proposition 2.

**Proof of Proposition 2 in the case $\mu_n = o(n^{2/3})$.** Let $q_1, \cdots, q_u$ be such that $\#\{i, q_i = 1\} = s$. Then $\sum_{\{i,q_i \geq 2\}} (q_i - 2) = q - 2u + s$. Since $\sum_{k=1}^K \beta_{n,k}^2 = 1$, we have, by definition of $\mu_n$, $\sum_{k=1}^K |\beta_{n,k}| = O(\mu_n^{1/2})$. Thus,

$$\sum\nolimits'' |\beta_{n,k_1}^{q_1} \cdots \beta_{n,k_u}^{q_u}| \leq C\mu_n^{s/2} b_n^{q-2u+s}.$$

Since all terms involved in the expansion of $\mathbb{E}(\prod_{i=1}^u Y_{n,k_i}^{q_i})$ are of order $n^{-s/3}$ at most, we get, if $s > 0$,

$$\mathbb{E}(\prod_{i=1}^{u} Y_{n,k_i}^{q_i}) = O(\mu_n^{s/2} b_n^{q-2u+s} n^{-s/3}) = O((\mu_n/n^{2/3})^{s/2}) = o(1).$$

If $s = 0$ then either $u < q/2$ or $u = q/2$ and $q_1 = \cdots = q_u = 2$. In both cases, the condition $b_n = o(1)$ yields the required limit.

**Proof of Proposition 2 in the case $\tau \geq 4$.** Assume that for all $n$ and $1 \leq k \leq K$, the Hermite rank of $\psi_{n,k}$ is at least 4. This yields

$$A_n(q_1, \cdots, q_v) = \sum\nolimits_{v,n} \prod_{i=1}^{v} \beta_{n,k_i}^{q_i} \left\{ \prod_{i=1}^{v} \mathbb{E}[\psi_{n,k_i}^{q_i}(\xi)] + o(^{-s/2}) \right\}.$$

The expectation term above vanishes when the number ($s$) of indices $i$ such that $q_i = 1$ is not zero, we get, for such $v$-tuples,

$$|A_n(q_1, \cdots, q_v)| \leq C\epsilon_n n^{-s/2} \left( \sum_{k=1}^{K} |\beta_{n,k}| \right)^s b_n^{v-s},$$

where $\lim_{n\to\infty} \epsilon_n = 0$. Since $\sum_{k=1}^K \beta_{n,k}^2 = 1$, applying Hölder inequality, we have $\sum_{k=1}^K |\beta_{n,k}| = O(\sqrt{n})$, thus if $s \neq 0$, $A_n(q_1, \cdots, q_v) = o(1)$. If $s = 0$ and $v < q/2$, then as before $A_n(q_1, \cdots, q_v) = o(1)$. If $s = 0$ and $v = q/2$, then

$$A_n(2, \cdots, 2) = \sum\nolimits_{q/2,n} \prod_{i=1}^{q/2} \sigma_{n,k}^2 + o(1).$$

The proof is concluded as in the general case by noting that under the Lindeberg condition $b_n = o(1)$, $s_n^q = \sum_{q/2,n} \prod_{i=1}^{q/2} \sigma_{n,k}^2 + o(1)$.

6.2. **Proof of Lemma 1.** Define $\tilde\sigma_M^2 = \mathbb{E}(Z_t^2 \mathbf{1}_{\{|\mathbf{Z_t}|>\mathbf{M}\}})$ and $\tilde Z_t^{(M)} = \tilde\sigma_M^{-1} Z_t \mathbf{1}_{\{|\mathbf{Z_t}|>\mathbf{M}\}}$. Define $\tilde W_{n,k}^{(M)}$ in the obvious way wrt $\tilde Z_t^{(M)}$. With these notations, we have $W_{n,k} = \sigma_M W_{n,k}^{(M)} + \tilde\sigma_M \tilde W_{n,k}^{(M)}$.

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_{k=1}^K \beta_{n,k}\{\psi_{n,k}(W_{n,k}) - \psi_{n,k}(W_{n,k}^{(M)})\}\Big)^2\Big] \\
&= \sum_{k=1}^K \beta_{n,k}^2 \Big\{\mathbb{E}[\psi_{n,k}^2(W_{n,k})] + \mathbb{E}[\psi_{n,k}^2(W_{n,k}^{(M)})] - 2\mathbb{E}(\psi_{n,k}(W_{n,k})\psi_{n,k}(W_{n,k}^{(M)}))\Big\} \\
&+ \sum_{1\le k\ne l\le K} \beta_{n,k}\beta_{n,l}\mathbb{E}[\{\psi_{n,k}(W_{n,k}) - \psi_{n,k}(W_{n,k}^{(M)})\}\{\psi_{n,l}(W_{n,l}) - \psi_{n,l}(W_{n,l}^{(M)})\}] =: A_{n,M} + B_{n,M}.
\end{aligned}$$

Let $\xi^{(1)}$ and $\xi^{(2)}$ be two independent $2m$-dimensional standard Gaussian vectors. As shown in section 8.1 below, we can apply Theorem 3.17 in Götze and Hipp (1978) [9] and we get

$$\begin{aligned}
&\mathbb{E}[\psi_{n,k}^2(W_{n,k})] = \mathbb{E}[\psi_{n,k}^2(\xi^{(1)})] + O(n^{-1/2}), \\
&\mathbb{E}[\psi_{n,k}^2(W_{n,k}^{(M)})] = \mathbb{E}[\psi_{n,k}^2(\xi^{(1)})] + O(n^{-1/2}), \\
&\mathbb{E}[\psi_{n,k}(W_{n,k})\psi_{n,k}(W_{n,k}^{(M)})] = \mathbb{E}[\psi_{n,k}(\xi^{(1)})\psi_{n,k}(\sigma_M\xi^{(1)} + \tilde\sigma_M\xi^{(2)})] + O(n^{-1/2}),
\end{aligned}$$

where the bounds are uniform because of (10). These expansions yield

$$A_{n,M} = \sum_{k=1}^K \beta_{n,k}^2 \mathbb{E}(\psi_{n,k}(\xi^{(1)})\{\psi_{n,k}(\sigma_M\xi^{(1)} + \tilde\sigma_M\xi^{(2)}) - \psi_{n,k}(\xi^{(1)})\}) + O(n^{-1/2}).$$

Applying Hölder inequality, (10) and the mean value theorem, we get, for some constant $C$ that does not depend on $M$,

$$A_{n,M} \le C\mathbb{E}^{1/2}[(|\sigma_M\xi^{(1)} + \tilde\sigma_M\xi^{(2)}|^2 - |\xi|^2)^2] + O(n^{-1/2}).$$

Thus $\limsup_n A_{n,M} \le \mathbb{E}^{1/2}((|\sigma_M\xi^{(1)} + \tilde\sigma_M\xi^{(2)}|^2 - |\xi^{(1)}|^2)^2]$, and since $\lim_{M\to\infty}\sigma_M = 1$, by the bounded convergence theorem, we get $\lim_{M\to\infty}\limsup_n A_{n,M} = 0$.

To deal with the second term $B_{n,M}$ we need to use an Edgeworth expansion up to the order $n^{-1}$. Applying again Theorem 3.17 in Götze and Hipp (1978), with assumption (10), we get

$$\sum_{1\le k\neq l\le K} \beta_{n,k}\beta_{n,l}\mathbb{E}[\psi_{n,k}(W_{n,k})\psi_{n,l}(W_{n,l})]$$
$$= \frac{\kappa_4}{4n}\sum_{1\le k_1\neq k_2\le K}\beta_{n,k_1}\beta_{n,k_2}\sum_{j_1,j_2=1\cdots 2m} C_2(\psi_{n,k_1},j_1)C_2(\psi_{n,k_2},j_2)+o(1),$$
$$\sum_{1\le k\neq l\le K} \beta_{n,k}\beta_{n,l}\mathbb{E}[\psi_{n,k}(W_{n,k})\psi_{n,l}(W_{n,l}^{(M)})]$$
$$= \frac{\kappa_4^{(M)}-\sigma_M^2\tilde{\sigma}_M^2}{4n}\sum_{1\le k\neq l\le K}\beta_{n,k}\beta_{n,l}\sum_{j_1,j_2=1\cdots 2m} C_2^M(\psi_{n,k_1},j_1)C_2(\psi_{n,k_2},j_2)+o(1),$$

where $\kappa_4^{(M)}$ is the fourth order cumulant of $Z_0^{(M)}$ and $C_2^{(M)}(\psi,j)=\mathbb{E}[H_2(\xi_j^{(1)})\psi(\sigma_M\xi^{(1)}+\tilde{\sigma}_M\xi^{(2)})]$. Under (10), the coefficients $C_2(\psi_{n,k},j)$ and $C_2^M(\psi_{n,k},j)$ are uniformly bounded and thus we get, applying Hölder inequality and the mean value theorem,

$$\limsup_n B_{n,M}\le C\,\left(\sigma_M^2\tilde{\sigma}_M^2+|\kappa_4-\kappa_4^{(M)}|\right).$$

As for $A_{n,M}$, we conclude by applying the bounded convergence theorem.

6.3. **Proof of Lemma 2.** Under the assumptions of Lemma 2, using Lemma 3, it is easily seen that for all $1\le k\neq j\le K$, the following expansions are valid.

$$\mathbb{E}[\psi_{n,k}^2(W_{n,k})]=\|\psi_{n,k}\|^2+O(n^{-1/2}),$$
$$\mathbb{E}[\psi_{n,k}(W_{n,k})\psi_{n,k}(W_{n,k})]=\frac{\kappa_4}{4n}\sum_{1\le i_1,i_2,\le 2m}C_2(\psi_{n,k},i_1)C_2(\psi_{n,j},i_2)+n^{-1/2}\mathbb{F}(\psi_{n,k},\psi_{n,j})+o(n^{-1}),$$
$$|\mathbb{F}(\psi_{n,k},\psi_{n,j})|\le C\|\psi_{n,k}\|\|\psi_{n,j}\|\Delta(k,j),$$

where $\Delta$ vanishes outside a subspace of $\mathbb{N}^2$ of dimension at most 1, and the terms $O(n^{-1/2})$ and $o(n^{-1})$ are uniform because of the assumptions of Lemma 2. Summing these expressions yields Lemma 2.

## 7. Proof of Theorem 3

We need only prove the tightness of the sequence $\xi_n(x):=\sqrt{K}\{\hat{F}_n(x)-F_m(x)\}$ on a compact set $[0,M]$. For that we must compute the moments of $\xi_n(x)-\xi_n(y)$ for some $0\le x<y\le M$. Denote $\psi_{x,y}(t)=$

$\mathbf{1}_{\{\mathbf{x}<\mathbf{t}\leq\mathbf{y}\}} - (\mathbf{F_m}(\mathbf{y}) - \mathbf{F_m}(\mathbf{x}))$. Let $q$ be a positive integer and let $m_{n,q}(x,y) = \mathbb{E}[(\xi_n(x) - \xi_n(y))^{2q}]$. Using the same notations as in the proof of Proposition 2, we have the expansion

$$m_{n,q}(x,y) = \sum_{v=1}^{q} \sum\nolimits'_{v,q} \frac{q!}{q_1! \cdots q_v!} \frac{1}{v!} A_n(q_1, \cdots, q_v),$$

$$A_n(q_1, \cdots, q_v) = n^{-q/2} \sum\nolimits''_{v,n} \mathbb{E}\left[\prod_{i=1}^{v} \psi_{x,y}^{q_i}(2\pi \bar{I}_{n,k_i})\right].$$

We now use Lemma 3 to obtain an expansion of the expectation above under Assumption **(A1)**. Denote $C_m(x,y)$ Assuming $Z_0$ has enough finite moments, we get

$$\mathbb{E}\left[\prod_{i=1}^{v} \psi_{x,y}^{q_i}(2\pi \bar{I}_{n,k_i})\right] = n^{-s/2} \frac{s!(m^2 C_m^2(x,y)\kappa_4/2)^{s/2}}{(s/2)!} \prod_{\{j, q_j \geq 2\}} \mathbb{E}[\psi_{x,y}^{q_j}(\xi)] \mathbf{1}_{\{\mathbf{s}\in\mathbf{2}\mathbb{N}\}}$$
$$+ \sum_{r=[(2s+2)/3]}^{s} n^{-r/2} \mathbb{F}_{r,k}(\psi_{x,y}^{q_1}, \cdots, \psi_{x,y}^{q_v}) + n^{-s/2} r_n(\psi_{x,y}^{q_1}, \cdots, \psi_{x,y}^{q_v}, k),$$

We must now bound all these terms by powers of $y - x$. It is easily seen that there exists a constant $C$ such that $|C_m(x,y)| \leq C(y-x)$ and $|\mathbb{E}[\psi_{x,y}^{q_j}(\xi)]| \leq C(y-x)$. Thus the first term is bounded by $n^{-s/2}(y-x)^v$. Since it also holds that $\|\psi_{x,y}^q\|^2 \leq C(y-x)$ and $N_\alpha(\psi_{x,y}^q) \leq C(y-x)$ for any positive integer $q$, we get

$$\sum_{r=[(2s+2)/3]}^{s} n^{-r/2} \mathbb{F}_{r,k}(\psi_{x,y}^{q_1}, \cdots, \psi_{x,y}^{q_v}) \leq C \sum_{r=[(2s+2)/3]}^{s} n^{-r/2} \Delta_r(k)(y-x)^v \leq C n^{v-s/2}(y-x)^v,$$

$$n^{-s/2} r_n(\psi_{x,y}^{q_1}, \cdots, \psi_{x,y}^{q_v}, k) \leq C n^{-(s+1)/2}(y-x)^v.$$

Altogether, we get

$$A_n(q_1, \cdots, q_v) \leq C n^{-q/2} n^{v-s/2}(y-x)^v = C n^{-q/2+v-s/2}(y-x)^v.$$

Since for a given $q$, $v$ is at least equal to one and at most equal to $q$, we get for $|y-x| \leq 1/n$,

$$m_{n,q}(x,y) \leq C(n^{-q/2+1}|x-y| + |y-x|^{q/2}).$$

If $|y-x| \geq 1/n$, then since $v \leq (q+s)/2$, it always holds that

$$m_{n,q}(x,y) \leq C n^{-(q+s)/2}(n|y-x|)^v \leq C n^{-(q+s)/2}(n|y-x|)^{(q+s)/2} \leq C|y-x|^{q/2}.$$

Finally, we get, for $q = 4$, provided that $\mathbb{E}[|Z_0|^8] < \infty$,

$$\mathbb{E}[(\xi_n(x) - \xi_n(y))^4] \leq C(n^{-1}|x-y| + |y-x|^2).$$

This ensures the tightness of the empirical spectral process.

## 8. Proof of Lemma 3

Let $k = (k_1, \cdots, k_d)$ be a $d$-tuple of pairwise distinct integers. Let $\xi^{(1)}, \cdots, \xi^{(d)}$ be $d$ independent $2m$-dimensional standard Gaussian vectors and denote $\boldsymbol{\xi} = (\xi^{(1)}, \cdots, \xi^{(d)})^T$. Denote $\psi(\boldsymbol{\xi}) = \prod_{j=1}^d \phi_j(\xi^{(j)})$. In section 8.1, it will be proved that assumptions **(BR)** and **(GH)** imply respectively that the assumptions of Theorem 19.4 in Bhattacharya and Rao (1976) and Theorem 3.17 in Götze and Hipp (1978) hold. Then

$$\mathbb{E}[\phi_1(W_{n,k_1}) \cdots \phi_d(W_{n,k_d})] = \sum_{r=0}^{\mu^*} n^{-r/2} \mathbb{E}_{r,k}(\phi_1, \cdots, \phi_d) + n^{-s/2} \epsilon_n R_n(\phi_1, \cdots, \phi_d), \tag{16}$$

where $\epsilon_n$ depends only on the distribution of $Z_0$ and $\alpha^*$ and verifies $\lim_{n\to\infty} \epsilon_n = 0$, and

- under assumption **(BR)**, $\mu^* = \alpha_1 + \cdots + \alpha_d$ and
$$|R_n(\phi_1, \cdots, \phi_d)| \le \epsilon_n \prod_{i=1}^d N_{\alpha_i}(\phi_i),$$
- under assumption **(GH)**, $\mu^* = \nu - 2$ and
$$|R_n(\phi_1, \cdots, \phi_d)| \le C \prod_{i=1}^d M_{\nu_i, r}.$$

If $\mu^* > s$, then we must first prove that the terms $\mathbb{E}_{r,k}$ can be conveniently bounded for $s + 1 \le r \le \mu^*$. Let us first give explicit expressions for the quantities $\mathbb{E}_{r,k}$. They derive from the formal Edgeworth expansion up to the order $s$ of $\mathbb{E}[\prod_{i=1}^d \phi_i(W_{n,k_i})]$, which is the same under both sets of assumptions.

$$\mathbb{E}_{r,k}(\phi_1, \cdots, \phi_d) = \sum_{t=1}^{r} \frac{1}{t!} {\sum}^*_{r,t} \frac{\chi_{\nu_1}(k) \cdots \chi_{\nu_t}(k)}{\nu_1! \cdots \nu_t!} \mathbb{E}[H_{\nu_1+\cdots+\nu_t}(\boldsymbol{\xi}) \psi(\boldsymbol{\xi})], \tag{17}$$

- ${\sum}^*_{r,t}$ extends over all $t$-tuples of multi-indices $\nu_1, \cdots, \nu_t$ such that $\nu_i = (\nu_i(1), \cdots, \nu_i(2md)) \in \mathbb{N}^{2md}$,
$$|\nu_i| = \nu_i(1) + \cdots + \nu_i(2md) \ge 3, \quad \sum_{i=1}^t |\nu_i| = r + 2t,$$
- for $\nu \in \mathbb{N}^{2md}$, $\nu! = \prod_{j=1}^{2md} \nu(j)!$
- $H_\nu$ is a multidimensional Hermite polynomial, *i.e.* $H_\nu(\xi) = H_{\nu(1)}(\xi_1) \cdots H_{\nu(2md)}(\xi_{2md})$,

- for $k \in \{1, \cdots, K\}^{md}$ and $\nu \in \mathbb{N}^{2md}$, $\chi_\nu(k)$ is the following cumulant

$$\chi_\nu(k) = 2^{|\nu|/2} \kappa_{|\nu|} A_\nu(k), \tag{18}$$

$$A_\nu(k) = n^{-1} \sum_{t=1}^{n} \prod_{j=1}^{d} \prod_{i=1}^{m} \cos(t x_{m(k_j-1)+i})^{\nu_{2m(j-1)+2i-1}} \sin(t x_{m(k_j-1)+i})^{\nu_{2m(j-1)+2i}}, \tag{19}$$

where $\kappa_{|\nu|}$ is the cumulant of order $|\nu|$ of $Z_0$.

Clearly, $|A_\nu| \le 1$, thus, for $0 \le r \le \mu^*$, there exists a constant $C_d$, uniform wrt $n$ and $k = (k_1, \cdots, k_d)$ such that

$$|\mathbb{E}_{r,k}(\phi_1, \cdots, \phi_d)| \le C_d \prod_{i=1}^{d} \|\phi_i\|.$$

Thus, under both assumptions, if $\mu^* > s$, we have

$$\left| \sum_{r=s+1}^{\mu^*} n^{-r/2} \mathbb{E}_{r,k}(\phi_1, \cdots, \phi_d) \right| \le C n^{-(s+1)/2} \prod_{i=1}^{d} \|\phi_i\|,$$

for some constant which depends only on the distribution of $Z_0$ and $\mu^*$. Since $\|\phi\| \le C_\alpha N_\alpha(\phi)$ and $\|\phi\| \le C_{\nu,r} M_{\nu,r}(\phi)$, we finally get

$$\left| \mathbb{E}[\phi_1(W_{n,k_1}) \cdots \phi_d(W_{n,k_d})] - \sum_{r=}^{s} n^{-r/2} \mathbb{E}_{r,k}(\phi_1, \cdots, \phi_d) \right| \le n^{-s/2} \epsilon_n r_n(\phi_1, \cdots, \phi_d),$$

where $r_n$ satisfies either (14) or (15). We can now consider the terms $\mathbb{E}_{r,k}(\phi_1, \cdots, \phi_d)$ for $1 \le r \le s$. The desired properties of these terms derive from considerations on moments and on cumulants.

**Moments.** We need only consider the $t$-tuples of multi-indices $\underline{\nu} = (\nu_1, \cdots, \nu_t)$ such that

$$\mathbb{E}[H_{\nu_1 + \cdots + \nu_t}(\boldsymbol{\xi}) \psi(\boldsymbol{\xi})] \ne 0. \tag{20}$$

Let $\tau_i$ be the Hermite rank of $\phi_i$ $1 \le i \le s$, and recall that $\tau = \inf\{\tau_i, i = 1, \cdots, s\}$. By definition of the Hermite rank, we get

$$\text{If } |\nu_1 + \cdots + \nu_t| < s\tau, \quad \mathbb{E}[H_{\nu_1 + \cdots + \nu_t}(\boldsymbol{\xi}) \psi(\boldsymbol{\xi})] = 0. \tag{21}$$

The definition of $\sum_{r,t}^{*}$, implies that $t \le r$ and $|\nu_1 + \cdots + \nu_t| = r + 2t \le 3r$, thus, $3r$ must be greater than or equal to $s\tau$ for the term $\mathbb{E}_{r,k}(\phi_1, \cdots, \phi_d)$ not to vanish. If $\tau \ge 4$, this implies that all terms $\mathbb{E}_{r,k}$ are vanishing for $r \leq s$ and (13) is proved. If $\tau = 2$, we must study all terms $2s/3 \leq r \leq s$. Since the $\xi_i$'s are i.i.d. standard Gaussian, the expectation in (21) writes

$$\mathbb{E}[H_{\nu_1 + \cdots \nu_t}(\xi)\psi(\xi)] = \prod_{i=1}^{d} \mathbb{E}[\prod_{j=1}^{2m} H_{\nu_1(2m(i-1)+j)+\cdots+\nu_t(2m(i-1)+j)}(\xi_{2m(i-1)+j})\phi_i(\boldsymbol{\xi}^{(i)})].$$

Since $\tau_i \geq 2$ for all $1 \leq i \leq s$, (20) implies the following condition

$$\sum_{j=1}^{2m}\sum_{l=1}^{t} \nu_l(2m(i-1)+j) \geq 2, \quad 1 \leq i \leq s; \tag{22}$$

**Cumulants.** The well known orthogonality properties of the functions sine and cosine computed at the Fourier frequencies imply that $A_\nu$ writes

$$A_\nu(k) = 2^{-|\nu|/2} + \delta_\nu(k), \tag{23}$$

if all the components of $\nu$ are even, ($\nu \in (2\mathbb{N})^{2md}$), and

$$A_\nu(k) = \delta_\nu(k), \tag{24}$$

if at least one of the component of $\nu$ is odd, $\delta_\nu$ having following properties

1. $\delta_\nu$ depends only on $\nu$ ;
2. $\delta_\nu$ identically vanishes outside a finite union of strict hyperplanes of $\mathbb{R}^d$ ;
3. $\forall k \in \mathbb{N}^d, \quad |\delta_\nu(k)| \leq 1$ ;

Thus, for each $\nu$, $A_\nu(.)$ is constant outside a finite union of strict hyperplanes of $\{1, \cdots, K\}^d$. To illustrate these properties, we give two examples in the case $m = 1$, $d = 2$. Assume $n$ is even and let $\nu = (2, 0, 1, 1)$. Then $|\nu| = 4$ and

$$A_\nu(k) = n^{-1}\sum_{t=1}^{n} \cos^2(txk_1)\cos(txk_3)\sin(txk_4) = \frac{1}{8n}\sum_{t=1}^{n}(2\sin t(x_{k_3} + x_{k_4}) - 2\sin t(x_{k_3} - x_{k_4})$$
$$+ \sin t(2x_{k_1} + x_{k_3} + x_{k_4}) + \sin t(2x_{k_1} - x_{k_3} + x_{k_4}) + \sin t(-2x_{k_1} + x_{k_3} + x_{k_4}) + \sin t(-2x_{k_1} - x_{k_3} + x_{k_4})).$$

Thus $A_\nu(k) = 1/4 + \delta_\nu(k)$, where $\delta_\nu$ vanishes outside the sets $2k_1 + k_3 + k_4 = 0$, $2k_1 - k_3 + k_4 = 0$, $-2k_1 + k_3 + k_4 = 0$, $-2k_1 - k_3 + k_4 = 0$, $k_3 + k_4 = 0$ and $k_3 - k_4 = 0$, where the equalities must hold modulo $n$.

Let $\nu = (2,0,2,0)$, then $|\nu| = 4$ and

$$A_\nu(k) = n^{-1}\sum_{t=1}^n \cos^2(tx_{k_1})\cos^2(tx_{k_3}) =$$
$$\frac{1}{4} + \frac{1}{8n}\sum_{t=1}^n \left(2\cos(2tx_{k_1}) + 2\cos(2tx_{k_3}) + \cos(2t(x_{k_1}+x_{k_3})) + \cos(2t(x_{k_1}-x_{k_3}))\right).$$

Thus $A_\nu(k) = 1/4 + \delta_\nu(k)$, where $\delta_\nu$ vanishes outside the sets $2k_1 = 0, (\text{mod } n)$, $k_3 = 0, (\text{mod } n)$, $2k_3 = n, (\text{mod } n)$, $k_1 \pm k_3 = 0, (\text{mod } n)$.

We return to the general case. For each $\underline{\nu}$, the multi-indices $k$ such that

$$\prod_{i=1}^t \chi_{\nu_i}(k) \neq 0 \tag{25}$$

belong to a finite union of subspaces of $\mathbb{R}^d$, and denote $d(\underline{\nu})$ the greatest dimension of these subspaces. Lemma 3 will be proved as a consequence of the following fact

$$\text{for all } \underline{\nu} \text{ such that (20) holds, } d(\underline{\nu}) < d + (r-s)/2. \tag{26}$$

We now prove (26), first in the case $r = s$.

**Case $r = s$.** First note that

$$\exists i = 1, \cdots, t,\ \exists j = 1,\ \cdots, 2md,\ \nu_i(j) \notin 2\mathbb{N} \ \Rightarrow\ d(\underline{\nu}) < d.$$

Consider now the $t$-tuples of multi-indices $\nu_1, \cdots, \nu_t$ without any odd component. Since $|\nu_i|$ must be greater than or equal to 3 for all $i = 1, \cdots, t$, $|\nu_i| \geq 4$ necessarily holds. This implies that $r + 2t = |\nu_1 + \cdots + \nu_t| \geq 4t$, and thus $2t \leq r$ and $2s \leq r + 2t \leq 2r \leq 2s$. Finally, we conclude that $r = s$ and $4t = 2s$ which implies that $s$ must be even, and for all $i = 1, \cdots, s/2$, $|\nu_i| = 4$. Notice now that no single component of any of the $\nu_i$ can be equal to 4, otherwise (22) would not hold. Thus, if $\underline{\nu}$ is such that (20) holds, the following conditions must hold

$$\forall j \in \{1, \cdots, s\}, \exists! l_j \in \{2m(j-1)+1, \cdots, 2mj\}, \nu_1 + \cdots + \nu_{s/2}(l_j) = 2; \tag{27}$$

$$\forall j \in \{1, \cdots, s\}, \forall l \in \{2m(j-1)+1, \cdots, 2mj\} \setminus \{l_j\}, \nu_1 + \cdots + \nu_{s/2}(l) = 0; \tag{28}$$

$$\forall j \in \{s+1, \cdots, d\}, \forall l \in \{2m(j-1)+1, \cdots, 2mj\}, \nu_1 + \cdots + \nu_{s/2}(l) = 0. \tag{29}$$

For such a $\underline{\nu}$, there exist integers $j_1, \cdots, j_s$ in $\{1, \cdots, 2m\}$ such that

$$\mathbb{E}[H_{\nu_1+\cdots+\nu_t}(\boldsymbol{\xi})\psi(\boldsymbol{\xi})] = \prod_{j=1}^{s} C_2(\phi_i, j_i) \prod_{j=s+1}^{d} \mathbb{E}[\phi_j(\xi^{(j)})].$$

Conversely, each $s$-tuple $j_1, \cdots, j_s$ is obtained by exactly $2^{-s/2}s!$ $\underline{\nu}$'s. Thus if $s$ is even, we get

$$\mathbb{E}_{s,k}(\phi_1, \cdots, \phi_d) = \frac{s!\kappa_4^{s/2}}{(s/2)!2^{3s/2}} \left[ \sum_{j_1,\cdots,j_s=1\cdots 2m} \prod_{i=1}^{s} C_2(\phi_i, j_i) \right] \prod_{i=s+1}^{d} \mathbb{E}[\phi_j(\boldsymbol{\xi}^{(i)})] + \mathbb{F}_{s,k}(\phi_1, \cdots \phi_d),$$

and if $s$ is odd, $\mathbb{E}_{s,k}(\phi_1, \cdots, \phi_d) = \mathbb{F}_{s,k}(\phi_1, \cdots \phi_d)$ where, in both cases, $\mathbb{F}_{s,k}$ has the claimed properties.

**Case $r < s$.** If $r < s$, then there is at least one index $i$ such that $|\nu_i| = 3$. Let $l(\underline{\nu})$ be the number of indices $i$ such that $\nu_i$ has at least one odd component. Since $r + 2t \geq 2s$, let $r + 2t = 2s + q$. By definition of $l(\underline{\nu})$, we have $r + 2t \geq 3l(\underline{\nu}) + 4(t - l(\underline{\nu}))$, whence $r + l(\underline{\nu}) \geq 2t$. Since $r + 2t = 2s + q$ and $r < s$, we get for any $d \geq s$,

$$r + l(\underline{\nu}) \geq 2t = 2s + q - r > s + q,$$

$$l(\underline{\nu}) - q > s - r,$$

$$d - (l(\underline{\nu}) - q)/2 < d + (r - s)/2. \tag{30}$$

The proof of Lemma 3 will be concluded if we prove the following bound :

$$d(\underline{\nu}) \leq d - [(l(\underline{\nu}) - q)/2 \vee 1]. \tag{31}$$

**Proof of (31).** To prove (31), we momentarily forget any reference to moments conditions. Denote $m(\underline{\nu}) = d - d(\underline{\nu})$. $m(\underline{\nu})$ is the minimum codimension of any subspace of $\mathbb{N}^{md}$ on which $A_{\underline{\nu}}(.)$ does not identically vanish. Following Velasco (1997) [13], $m(\underline{\nu})$ will be called the NRES, *i.e.* the minimum number of linear restrictions necessary to make the cumulants considered different from zero. The NRES $m(\underline{\nu})$ is obviously a non decreasing function of the number of indices $i$ such that $\nu_i$ has at least one odd component, as can be seen from (24) and the properties of $\delta_\nu$.

**(i).** Our first argument is that if there are at most two odd component in any single column of the array $\underline{\nu}$, then $m(\underline{\nu})$ is at least equal to $l(\underline{\nu})/2$, since each line of the array (*i.e.* each $\nu_i$) with at least one odd component yields one restriction, and different lines will yield different restrictions, except if their odd components are in the same columns. Thus (31) holds in this case.

**(ii).** If there exists at least one column with at least three odd components, let $z(\underline{\nu})$ denote the number of such columns and let $y(\underline{\nu})$ denote the total number of odd components in these columns. We now prove by induction on $y(\underline{\nu})$ that the following inequality holds :

$$m(\underline{\nu}) \geq (l(\underline{\nu}) - (y(\underline{\nu}) - 2z(\underline{\nu})))/2 \vee 1. \tag{32}$$

We have proved this property for $y(\underline{\nu}) = 0$, but we cannot start the induction at 0 since if $y(\underline{\nu}) \neq 0$, then $y(\underline{\nu}) \geq 3$. Thus we prove the property for $y(\underline{\nu}) = 3$, which implies $z(\underline{\nu}) = 1$. Let us first precisely define the induction assumption.

**Induction assumption.** Let $\underline{\nu}$ be an array of $t$ lines and $2d$ columns. Let $l$ be the number of *lines* which have at least one odd component. Let $z$ denote the number of *columns* which have at least three odd components and $y$ denote the number of odd components in these columns. Then $m(\underline{\nu}) \geq ((l - (y - 2z))/2) \vee 1$.

**Proof for $y = 3$.** In that case, $z = 1$ and we can cancel one line of the array in such a way as to obtain a new array $\nu'$ with $l' = l - 1$ and $y = z = 0$. For that array, we have $m' \geq l'/2$. So we have

$$m \geq m' \geq l'/2 = (l-1)/2 = (l - (y - 2z))/2.$$

**Induction.** Assume that the induction assumption is true for some $y - 1 \geq 3$. As above we cancel one of the line of the array and we obtain a new array $\nu'$ with $l' = l - 1$, $y' < y$ and $z' \leq z$. If $y' = 0$, then $m \geq m' \geq l'/2 = (l-1)/2 \geq (l - (y - 2z))/2$ since by definition $y \geq 3z$ and thus $y - 2z \geq 1$ as soon as $z \geq 1$. If $y' \neq 0$, then $y' \geq 3$ and we can apply the induction assumption. Thus we get

$$m \geq m' \geq (l' - (y' - 2z'))/2 = (l - (y' - 2z' + 1))/2.$$

Thus we must prove that $y' - 2z' + 1 \leq y - 2z$, *i.e.* $2(z - z') + 1 \leq y - y'$. If $z = z'$, this is obvious since $y' < y$. If $z' < z$, then $y - y' \geq 3(z - z') \geq 2(z - z') + 1$. This proves that the induction assumption holds for $y$.

**Conclusion.** We now prove that for an array which satisfies the moment conditions , we have $y(\underline{\nu}) - 2z(\underline{\nu}) \leq q$. Denote $w$ the the number of indices $j \in \{1, \cdots, d\}$ such that the sum of all the entries of the columns $2m(j-1)+1, \cdots, 2mj$ is exactly 1. Since the Hermite rank of $\phi_1, \cdots, \phi_s$ is at least 2, then it is necessary that $w \le d - s$, *i.e.* $d - w \ge s$. Thus, we have

$$2s + q = y + w + 2(d - z(\underline{\nu}) - w) = 2d - w + y - 2z(\underline{\nu}) \ge 2s + y - 2z(\underline{\nu}),$$

and thus $y - 2z(\underline{\nu}) \le q$. This, together with (32) yields $m(\underline{\nu}) \ge (l(\underline{\nu}) - q)/2 \vee 1$ and, by definition of $m(\underline{\nu})$, applying (30),

$$d(\underline{\nu}) \le d - ((l(\underline{\nu}) - q)/2) \vee 1 < d + (r - s)/2.$$

This concludes the proof of Lemma 3.

8.1. **Validity of Edgeworth expansions.** In this section, we prove that the Edgeworth expansions used in the previous sections are valid. Chen and Hannan (1980) [5] (Lemma 2) have adapted Theorem 19.3 of Bhattacharya and Rao (1976) [2] to prove that under assumption **(A1)**, the Edgeworth expansion of the joint density of an arbitrary number of discrete Fourier transform is valid up to the order 2. That was all they needed since they considered the function log and were only proving consistency of their estimator. To consider more genral functions, we should check the validity of the expansion up to an arbitrary order. We will omit this proof since the arguments of Chen and Hannan (1980) are easily generalized. We will only check the validity of Edgeworth expansions of moments using the result of Götze and Hipp (1978) [9]. We first state a version of Theorem 3.17 in Götze and Hipp (1978) [9] with stronger assumptions than in Götze and Hipp (1978), but which are easy to check in our context. Let $(\zeta_{n,k})_{1 \le k \le n}$ be a sequence of independent $a$-dimensional vectors. Define $S_n = n^{-1} \sum_{k=1}^{n} \zeta_{n,k}$ and let $Q_s$ be the formal Edgeworth expansion of $S_n$ up to the order $s$. Denote $|u|$ the Euclidean norm of a vector and define, whenever possible,

$$\rho_{n,3} = n^{-1} \sum_{k=1}^{n} \mathbb{E}(|\zeta_{n,k}|^3), \tag{33}$$

$$\Delta_{n,s} = n^{-1} \sum_{k=1}^{n} \mathbb{E}\big[|\zeta_{n,k}|^s \big\{n^{-1/2} |\zeta_{n,k}| \mathbf{1}_{\{|\zeta_{n,k}| \le n^{1/2}\}} + \mathbf{1}_{\{|\zeta_{n,k}| > n^{1/2}\}}\big\}\big]. \tag{34}$$

**Theorem 4.** *Let $\psi$ be a $C^{r+2}$ function on $\mathbb{R}^a$ and $p$ be an integer such that for all $\beta \in \mathbb{N}^a$ with $\sum_{i=1}^{a} \beta_i = r + 2$, it holds that*

$$\int_{-\infty}^{+\infty} \frac{|D^\beta \psi(x)|}{1 + |x|^p} dx \le C(\psi),$$

*for some finite constant $C(\psi)$. Assume that the variables $\zeta_{n,k}$ have finite moment of order $s+2$. If $\lim_{n\to\infty}\Delta_{n,s+2}=0$, then there exists a constant $C$ which depends only on $a$ and the distribution of $Z_0$ such that, for large enough $n$,*

$$\left|\mathbb{E}(\psi(S_n)) - Q_s(\psi)\right| \le C(M_s(\psi)+C(\psi))\Delta_{n,s+2}n^{-s/2} + C\rho_{n,3}^{r+k+1}n^{-(r+a+1)/2}).$$

In particular, if $\rho_{n,3}$ is uniformly bounded, then $\mathbb{E}(\psi(S_n))-Q_s(\psi)=o(n^{-s/2})$ as soon as $\psi$ is $C^{(s-a)^++2}$ and the constants involved in the term $o(n^{-s/2})$ depend only on the derivative of order $(s-a)^++2$. We now check that $\rho_{n,3}$ is bounded and $\lim_{n\to\infty}\Delta_{n,s+2}=0$ in our context. For $k=(k_1,\cdots,k_u)$, define

$$\zeta_{n,t} = \sqrt{2}Z_t(\cos(tx_{m(k_1-1)+1}),\sin(tx_{m(k_1-1)+1}),\cdots,\cos(tx_{mk_u}),\sin(tx_{mk_u}))^T.$$

Then $|\zeta_{n,t}|^2 = 2um|Z_t|^2$. In the context of Lemma 1, we must also consider

$$\begin{aligned}\zeta_{n,t}^{(M)} = \sqrt{2}(&Z_t^{(M)}\cos(tx_{m(k_1-1)+1}), Z_t^{(M)}\sin(tx_{m(k_1-1)+1}),\cdots, Z_t^{(M)}\cos(tx_{mk_1}), Z_t^{(M)}\sin(tx_{mk_1}),\\ &\tilde Z_t^{(M)}\cos(tx_{m(k_2-1)+1}), \tilde Z_t^{(M)}\sin(tx_{m(k_2-1)+1}),\cdots, \tilde Z_t^{(M)}\cos(tx_{mk_2}), \tilde Z_t^{(M)}\sin(tx_{mk_2}))^T.\end{aligned}$$

In that case we have

$$|\zeta_{n,t}^{(M)}|^2 = m(|Z_t^{(M)}|^2 + |\tilde Z_t^{(M)}|^2 = m|Z_t|^2(\sigma_M^{-2}\mathbf{1}_{\{|\mathbf{Z_t}|\le \mathbf{M}\}} + \tilde\sigma_{\mathbf{M}}^{-\mathbf{2}}\mathbf{1}_{\{|\mathbf{Z_t}|>\mathbf{M}\}}).$$

Thus, in both cases, $\rho_{n,3}$ is bounded, and $\lim_{n\to\infty}\Delta_{n,s+2}=0$ as soon as

$$\lim_{n\to\infty} n^{-1}\sum_{t=1}^n \mathbb{E}[|Y_t|^s\mathbf{1}_{\{|\mathbf{Y_t}|>\mathbf{n^{1/2}}\}}] = \mathbf{0} \quad \text{and} \quad \lim_{\mathbf{n}\to\infty}\mathbf{n^{-3/2}}\sum_{\mathbf{t=1}}^{\mathbf{n}}\mathbb{E}[|\mathbf{Y_t}|^{\mathbf{s+1}}\mathbf{1}_{\{|\mathbf{Y_t}|\le\mathbf{n^{1/2}}\}}] = \mathbf{0}$$

for an i.i.d. sequence $(Y_t)$ with finite moment of order $s$. This is obvious since the variables $Y_t$ are identically distributed, thus these sums are equal respectively to $\mathbb{E}[|Y_1|^s\mathbf{1}_{\{|\mathbf{Y_1}|>\mathbf{n^{1/2}}\}}]$ and $n^{-1/2}\mathbb{E}[|Y_1|^{s+1}\mathbf{1}_{\{|\mathbf{Y_1}|\le\mathbf{n^{1/2}}\}}]$. Since $|Y_1|^s\mathbf{1}_{\{|\mathbf{Y_1}|>\mathbf{n^{1/2}}\}}$ and $n^{-1/2}\mathbb{E}[|Y_1|^{s+1}\mathbf{1}_{\{|\mathbf{Y_1}|\le\mathbf{n^{1/2}}\}}]$ converge almost surely to 0 and both sequences are bounded for all $n$ by $|Y_1|^s$, their expectations tend to 0 as $n$ tends to infinity by the bounded convergence theorem.

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